Product Cordial Labeling in the Context of Tensor Product of Graphs

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Received: March 10, 2011 Accepted: March 28, 2011 doi:10.5539/jmr.v3n3p83

Abstract

For the graph G_1 and G_2 the tensor product is denoted by $G_1(T_p)G_2$ which is the graph with vertex set $V(G_1(T_p)G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1(T_p)G_2) = \{(u_1, v_1), (u_2, v_2)/u_1u_2 \epsilon E(G_1)$ and $v_1v_2 \epsilon E(G_2)\}$. The graph $P_m(T_p)P_n$ is disconnected for $\forall m, n$ while the graphs $C_m(T_p)C_n$ and $C_m(T_p)P_n$ are disconnected for both m and n even. We prove that these graphs are product cordial graphs. In addition to this we show that the graphs obtained by joining the connected components of respective graphs by a path of arbitrary length also admit product cordial labeling.

Keywords: Cordial labeling, Porduct cordial labeling, Tensor product

AMS Subject classification (2010): 05C78.

1. Introduction

We begin with simple, finite and undirected graph $G = (V(G), E(G))$. For standard terminology and notations we follow (West, D. B, 2001). The brief summary of definitions and relevant results are given below.

1.1 Definition: If the vertices of the graph are assigned values subject to certain condition(s) then it is known as *graph labeling*.

1.2 Definition: A mapping $f : V(G) \rightarrow \{0, 1\}$ is called *binary vertex labeling* of G and $f(v)$ is called the *label* of vertex v of G under f.

For an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by $f^*(e = uv) = |f(u) - f(v)|$. Let $v_f(0), v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0), e_f(1)$ be the number of edges of G having labels 0 and 1 respectively under f^*.

1.3 Definition: A binary vertex labeling of graph G is called a *cordial labeling* if $|v_f(0)-v_f(1)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$. A graph G is called *cordial* if admits cordial labeling.

The concept of cordial labeling was introduced by (Cahit,1987, p.201-207) and in the same paper he investigated several results on this newly introduced concept.

Motivated through cordial labeling the concept of product cordial labeling was introduced in (Sundaram, M., Ponraj, R. and Somsundaram, S., 2004, p.155-163) which has the flavour of cordial lableing but absolute difference of vertex labels is replaced by product of vertex labels.

1.4 Definition: A binary vertex labeling of graph G with induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ defined by $f^*(e = uv) = f(u)f(v)$ is called a *product cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is *product cordial* if it admits product cordial labeling.

In (Sundaram, M., Ponraj, R. and Somsundaram, S., 2004, p.155-163) it has been proved that trees, unicyclic graphs of odd order, triangular snakes, dragons, helms and union of two path graphs are product cordial. They also proved that a graph with p vertices and q edges with $p \geq 4$ is product cordial then $q < \frac{p^2-1}{4}$.

The graphs obtained by joining apex vertices of k copies of stars, shells and wheels to a new vertex are proved to be product cordial in (Vaidya, S. K. and Dani, N. A., 2010, p.62-65). The product cordial labeling for some cycle related graphs is discussed in (Vaidya, S. K. and Kanani, K. K., 2010, p.109-116). In the same paper they have investigated product cordial labeling for the shadow graph of cycle C_n.

1.5 Definition: The *tensor product* of two graphs G_1 and G_2 denoted by $G_1(T_P)G_2$ has vertex set $V(G_1(T_P)G_2) = V(G_1) \times V(G_2)$ and the edge set $E(G_1(T_P)G_2) = \{(u_1, v_1)(u_2, v_2)/u_1u_2 \epsilon E(G_1)$ and $v_1v_2 \epsilon E(G_2)\}$.

We investigate some new results on product cordial labeling in the context of tensor product of graphs.

Throughout this work the related graphs are vizulaized through MATGRAPH, which is a toolbox for working with simple graphs in MATLAB. This tool box is available free from http://www.ams.jhu.edu/~ers/matgraph.

2. Main Results

2.1 Theorem: $P_m(T_p)P_n$ is product cordial.

Proof: Let P_m and P_n be two paths of length $m-1$ and $n-1$ respectively. Let $G = P_m(T_P)P_n$. Denote the vertices of G as u_{ij} where $1 \le i \le m$, $1 \le j \le n$. We note that $|V(G)| = mn$ and $|E(G)| = 2(m-1)(n-1)$.

We define binary vertex labeling $f : V(G) \to \{0, 1\}$ as follows.

$$f(u_{ij}) = \begin{cases} 1 & ;\text{ if } i+j \text{ is even} \\ 0 & ;\text{ otherwise.} \end{cases}$$

In view of the above defined labeling pattern the graph G under consideration satisfies the conditions for product cordiality as shown in *Table 1*. Hence $P_m(T_p)P_n$ is product cordial.

2.2 Example: The product cordial labeling for $P_5(T_p)P_3$ is shown in *Figure 1*.

2.3 Theorem: $C_m(T_p)P_n$ is product cordial for both m and n even.

Proof: Let $G = C_m(T_p)P_n$ be the graph obtained by tensor product of C_m and P_n. Denote the vertices of G as u_{ij} where $1 \le i \le m$ and $1 \le j \le n$. We note that $|V(G)| = mn$ and $|E(G)| = 2m(n-1)$.

We define binary vertex labeling $f : V(G) \to \{0, 1\}$ as follows.

$$f(u_{ij}) = \begin{cases} 1 & ;\text{ if } i+j \text{ is even} \\ 0 & ;\text{ otherwise.} \end{cases}$$

In view of the above defined labeling pattern the graph G under consideration satisfies the conditions for product cordial labeling as shown in *Table 2*. Hence $C_m(T_p)P_n$ is product cordial for both m and n even.

2.4 Example: The product cordial labeling for $C_4(T_p)P_6$ is shown in *Figure 2*.

2.5 Theorem: $C_m(T_p)C_n$ is product cordial for m and n even.

Proof: Let C_m and C_n be the cycles with m and n vertices respectively. Let $G = C_m(T_p)C_n$ be the graph obtained by tensor product of C_m and C_n where m and n are even. Denote the vertices of G as u_{ij} where $1 \le i \le m$ and $1 \le j \le n$. We note that $|V(G)| = mn$ and $|E(G)| = 2mn$.

We define binary vertex labeling $f : V(G) \to \{0, 1\}$ as follows.

$$f(u_{ij}) = \begin{cases} 1 & ;\text{ if } i+j \text{ is even} \\ 0 & ;\text{ otherwise.} \end{cases}$$

In view of the above defined labeling pattern the graph G under consideration satisfies the conditions for product cordial labeling as shown in *Table 3*. Hence $C_m(T_p)P_n$ is product cordial for both m and n even.

2.6 Example: The product cordial labeling for $C_4(T_p)C_4$ is shown in *Figure 3*.

2.7 Theorem: The graph obtained by joining two components of $P_m(T_p)P_n$ with arbitrary path P_k is product cordial.

Proof: Let $G = P_m(T_p)P_n$ be the graph obtained by tensor product of P_m and P_n and G' be the graph obtained by joining two components of G by a path P_k. Let $u_1, u_2, \ldots, u_j$ and $v_1, v_2, \ldots, v_j$ respectively be the vertices of first and second component of G' where $j = \frac{mn}{2}$. Let $w_1, w_2, \ldots, w_k$ be the vertices of path P_k such that $u_1 = w_1$ and $v_1 = w_k$. We note that $|V(G')| = mn + k - 2$ and $|E(G')| = 2mn + k - 1$.

We define binary vertex labeling $f : V(G') \to \{0, 1\}$ as follows.

Case:1 $k \equiv 0(mod\, 2)$

$$f(u_i) = 0; \quad 1 \le i \le j$$
$$f(v_i) = 1; \quad 1 \le i \le j$$

$$f(w_i) = \begin{cases} 0; & 1 \le i \le \frac{k}{2} \\ 1; & \frac{k}{2} < i \le k \end{cases}$$

Case:2 $k \equiv 1 (mod\, 2)$

$$f(u_i) = 0; \quad 1 \le i \le j$$
$$f(v_i) = 1; \quad 1 \le i \le j$$

$$f(w_i) = \begin{cases} 0; & 1 \le i \le \left\lceil \frac{k}{2} \right\rceil \\ 1; & \left\lfloor \frac{k}{2} \right\rfloor \le i \le k \end{cases}$$

In view of the above defined labeling pattern f satisfies the conditions for product cordial labeling as shown in *Table 4*. Thus we prove that the graph obtained by joining two components of $P_m(T_p)P_n$ with arbitrary path P_k is product cordial for even m and n.

2.8 Example: The product cordial labeling for the graph obtained by joining two components of $P_5(T_p)P_3$ by path P_4 is shown in *Figure 4*.

2.9 Theorem: The graph obtained by joining two components of $C_m(T_p)P_n$ with arbitrary path P_k is product cordial for m and n even.

Proof: Let C_m be the cycle with m vertices, P_n be the path of length $n-1$ and G be the graph obtained by tensor product of C_m and P_n where m and n are even. Let G' be the graph obtained by joining two components of G by a path P_k and $u_1, u_2, \dots, u_j$ and $v_1, v_2, \dots, v_j$ be the vertices of first and second component of G' where $j = \frac{mn}{2}$. Let $w_1, w_2, \dots, w_k$ be the vertices of path P_k with $u_1 = w_1$ and $v_1 = w_k$. We note that $|V(G')| = mn + k - 2$ and $|E(G')| = 2m(n-1) + k - 1$.

We define binary vertex labeling $f : V(G') \to \{0, 1\}$ as follows.

Case:1 $k \equiv 0 (mod\, 2)$

$$f(u_i) = 0; \quad 1 \le i \le j$$
$$f(v_i) = 1; \quad 1 \le i \le j$$

$$f(w_i) = \begin{cases} 0; & 1 \le i \le \frac{k}{2} \\ 1; & \frac{k}{2} < i \le k \end{cases}$$

Case:2 $k \equiv 1 (mod\, 2)$

$$f(u_i) = 0; \quad 1 \le i \le j$$
$$f(v_i) = 1; \quad 1 \le i \le j$$

$$f(w_i) = \begin{cases} 0; & 1 \le i \le \left\lceil \frac{k}{2} \right\rceil \\ 1; & \left\lfloor \frac{k}{2} \right\rfloor \le i \le k \end{cases}$$

In view of the above defined labeling pattern the graph G' satisfies the conditions for product cordial labeling as shown in*Table 5*. That is, the graph obtained by joining two components of $C_m(T_p)P_n$ with arbitrary path P_k is product cordial for even m and n.

2.10 Example: The product cordial labeling for the graph obtained by joining two components of $C_4(T_p)P_6$ by path P_5 is shown in *Figure 5*.

2.11 Theorem: The graph obtained by joining two components of $C_m(T_p)C_n$ with arbitrary path P_k is product cordial for m and n even.

Proof: Let C_m and C_n be the cycle with m and n vertices respectively. Let G be the graph obtained by tensor product of C_m and C_n where m and n are even and G' be the graph obtained by joining two components of G by a path P_k. Let $u_1, u_2, \dots, u_j$ and $v_1, v_2, \dots, v_j$ respectively be the vertices of first and second component of G' where $j = \frac{mn}{2}$.

Let $w_1, w_2, \ldots, w_k$ be the vertices of path P_k with $u_1 = w_1$ and $v_1 = w_k$. We note that $|V(G')| = mn + k - 2$ and $|E(G')| = 2mn + k - 1$.

We define binary vertex labeling $f : V(G') \to \{0, 1\}$ as follows.

Case:1 $k \equiv 0(mod\,2)$

$$f(u_i) = 0; \quad 1 \le i \le j$$
$$f(v_i) = 1; \quad 1 \le i \le j$$
$$f(w_i) = \begin{cases} 0; & 1 \le i \le \frac{k}{2} \\ 1; & \frac{k}{2} < i \le k \end{cases}$$

Case:2 $k \equiv 1(mod\,2)$

$$f(u_i) = 0; \quad 1 \le i \le j$$
$$f(v_i) = 1; \quad 1 \le i \le j$$
$$f(w_i) = \begin{cases} 0; & 1 \le i \le \left\lceil \frac{k}{2} \right\rceil \\ 1; & \left\lfloor \frac{k}{2} \right\rfloor \le i \le k \end{cases}$$

In view of the above defined labeling pattern the graph G' under consideration satisfies the conditions for product cordial labeling as shown in *Table 6*. That is, the graph obtained by joining two components of $C_m(T_p)C_n$ with arbitrary path P_k is product cordial for even m and n.

2.12 Example: The product cordial labeling for the graph obtained by joining two components of $C_4(T_p)C_4$ by path P_4 is shown in *Figure 6*.

3. Concluding Remarks

As all the graphs are not product cordial graphs it is very interesting to investigate graphs or graph families which admit product cordial labeling. Here we investigate product cordial labeling for some graphs obtained by tensor product of two graphs. To derive similar results for other graph is an open area of research.

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Table 1.

	Vertex condition	Edge condition
m and n are odd	$v_f(0) + 1 = v_f(1) = \frac{mn+1}{2}$	$e_f(0) = e_f(1) = (m-1)(n-1)$
otherwise	$v_f(0) = v_f(1) = \frac{mn}{2}$	$e_f(0) = e_f(1) = (m-1)(n-1)$

Table 2.

	Vertex condition	Edge condition
m and n are even	$v_f(0) = v_f(1) = \frac{mn}{2}$	$e_f(0) = e_f(1) = m(n-1)$

Table 3.

	Vertex condition	Edge condition
m and n are even	$v_f(0) = v_f(1) = \frac{mn}{2}$	$e_f(0) = e_f(1) = mn$

Table 4.

	Vertex condition	Edge condition
k even	$v_f(0) = v_f(1) = \frac{mn+k-2}{2}$	$e_f(0) = e_f(1) + 1 = \frac{2(m-1)(n-1)+k}{2}$
k odd	$v_f(0) + 1 = v_f(1) = \frac{mn+k-1}{2}$	$e_f(0) = e_f(1) = \frac{2(m-1)(n-1)+k-1}{2}$

Table 5.

	Vertex condition	Edge condition
m and n are even, k even	$v_f(0) = v_f(1) = \frac{mn+k-2}{2}$	$e_f(0) = e_f(1) + 1 = \frac{2m(n-1)+k}{2}$
m and n are even, k odd	$v_f(0) + 1 = v_f(1) = \frac{mn+k-1}{2}$	$e_f(0) = e_f(1) = \frac{2m(n-1)+k-1}{2}$

Table 6.

	Vertex condition	Edge condition
m and n are even, k even	$v_f(0) = v_f(1) = \frac{mn+k-2}{2}$	$e_f(0) = e_f(1) + 1 = \frac{2mn+k}{2}$
m and n are even, k odd	$v_f(0) + 1 = v_f(1) = \frac{mn+k-1}{2}$	$e_f(0) = e_f(1) = \frac{2mn+k-1}{2}$

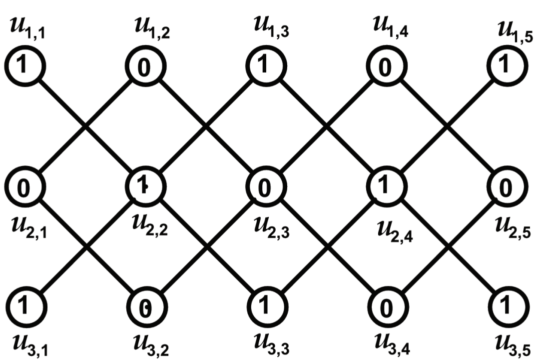

Figure 1. The product cordial labeling for $P_5(T_p)P_3$

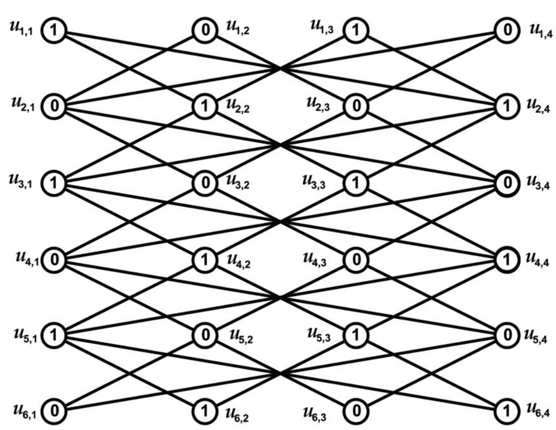

Figure 2. The product cordial labeling for $C_4(T_p)P_6$

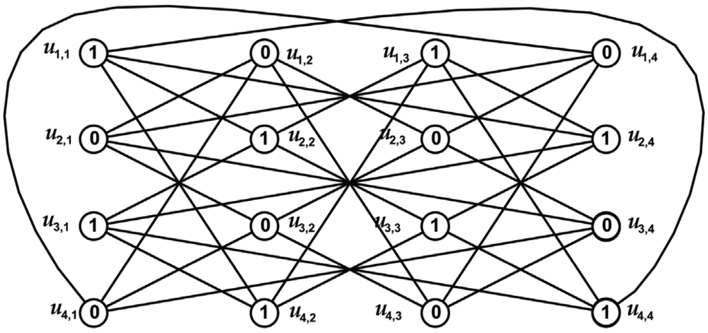

Figure 3. *The product cordial labeling for* $C_4(T_p)C_4$

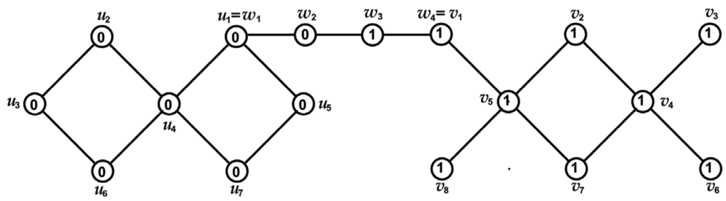

Figure 4. The product cordial labeling for the graph obtained by joining two components of $P_5(T_p)P_3$ by path P_4

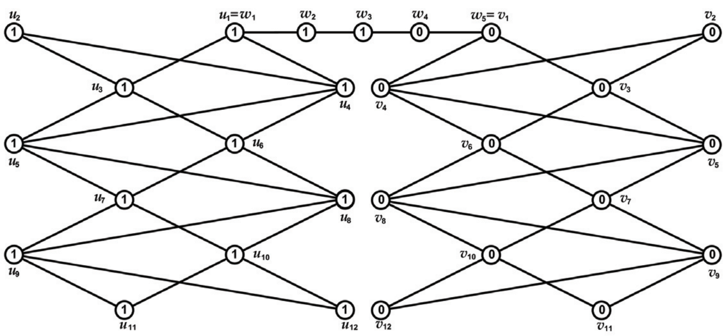

Figure 5. The product cordial labeling for the graph obtained by joining two components of $C_4(T_p)P_6$ by path P_5

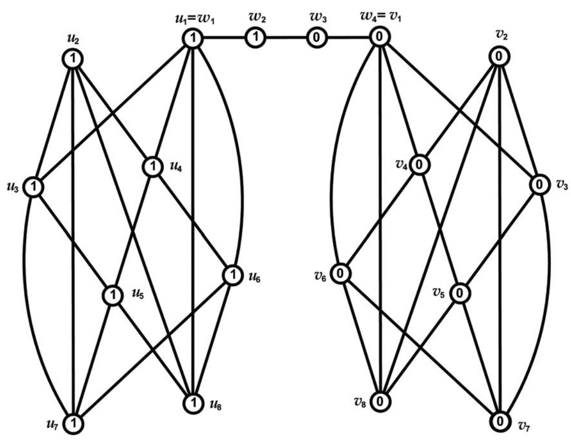

Figure 6. The product cordial labeling for the graph obtained by joining two components of $C_4(T_p)C_4$ by path P_4