Some Results on E-cordial Labeling

S. K. Vaidya and N. B. Vyas

***Abstract*—A binary vertex labeling $f : E(G) \rightarrow \{0, 1\}$ with induced labeling $f^* : V(G) \rightarrow \{0, 1\}$ defined by $f^*(v) = \sum\{f(uv) \mid uv \in E(G)\}(mod\, 2)$ is called E-cordial labeling of a graph G if the number of vertices labeled 0 and number of vertices labeled 1 differ by at most 1 and the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1. A graph which admits E-cordial labeling is called E-cordial graph. Here we prove that flower graph Fl_n, closed helm CH_n, double triangular snake DT_n and gear graph G_n are E-cordial graphs.**

***Index Terms*—Binary vertex labeling, Cordial labeling, E-cordial labeling, E-cordial graphs.**

***MSC 2010 Codes* - 05C78, 05C38.**

I. INTRODUCTION

We begin with finite, connected and undirected graph $G = (V(G), E(G))$ without loops and multiple edges. Throughout this paper $|V(G)|$ and $|E(G)|$ respectively denote the number of vertices and number of edges in G. For any undefined notation and terminology we rely upon Gross and Yellen [5]. In order to maintain compactness we will provide a brief summary of definitions and existing results.

Definition 1.1: A *graph labeling* is an assisgnment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (or edges) then the labeling is called a *vertex labeling* (or an *edge labeling*).

Beineke and Hegde[1] describe labeling of discrete structure as a frontier between graph theory and theory of numbers. For extensive survey of graph labeling as well as bibliographic references we refer Gallian[3].

Most of the graph labeling techniques trace their origin to graceful labeling introduced independently by Rosa[8] and Golomb[4] defined as follows.

Definition 1.2: A function $f : V(G) \rightarrow \{0, 1, \ldots, |E(G)|\}$ is called *graceful labeling* of graph G if f is injective and the induced function $f^*(e = uv) = |f(u) - f(v)|$ is bijective. A graph which admits graceful labeling is called a *graceful graph.*

The famous Ringel-Kotzig conjecture [7] and many illustrious work on it brought a tide of labeling problems with graceful theme.

Definition 1.3: A graph G is said to be *edge-graceful* if there exists a bijection $f : E(G) \rightarrow \{1, 2, \ldots, |E(G)|\}$ such that the induced mapping $f^* : V(G) \rightarrow \{0, 1, 2, \ldots, |V(G)| - 1\}$ given by $f^*(x) = \sum f(xy))(mod\, |V|)$, taken over all edges xy is a bijective.

The notion of edge gracefulness was introduced by Lo[6].

Definition 1.4: A mapping $f : V(G) \rightarrow \{0, 1\}$ is called *binary vertex labeling* of G and $f(v)$ is called the *label* of vertex v of G under f.

Notation: For an edge $e = uv$, the induced edge labeling $f^* : E(G) \rightarrow \{0, 1\}$ is given by $f^*(e = uv) = |f(u) - f(v)|$ then $v_f(i)$ = the number of vertices of G having label i under f and let $e_f(i)$ = the number of edges of G having label i under f^* for $i = 0, 1$.

Definition 1.5: A binary vertex labeling of graph G is called *cordial labeling* if $|v_f(0)-v_f(1)| \leq 1$ and $|e_f(0)-e_f(1)| \leq 1$. A graph G is called *cordial* if it admits cordial labeling.

The concept of cordial labeling was introduced by Cahit[2]. He also investigated several results on this newly defined concept.

Definition 1.6: Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ and let $f : E(G) \rightarrow \{0, 1\}$. Define f^* on $V(G)$ by $f^*(v) = \sum\{f(uv) \mid uv \in E(G)\}(mod\, 2)$. The function f is called an $E - cordial$ labeling of G if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph is called $E - cordial$ if it admits $E - cordial$ labeling .

In 1997 Yilmaz and Cahit[13] introduced E-cordial labeling as a weaker version of edge-graceful labeling and with the blend of cordial labeling. They proved that the trees with n vertices, K_n, C_n are E-cordial if and only if $n \not\equiv 2(mod\, 4)$ while $K_{m,n}$ admits E-cordial labeling if and only if $m + n \not\equiv 2(mod\, 4)$.

Vaidya and Lekha[9] have proved that the graphs obtained by duplication of an arbitrary vertex as well as an arbitrary edge in cycle C_n admit E-cordial labeling. In addition to this they show that the joint sum of two copies of cycle C_n, the split graph of even cycle C_n and the shadow graph of path P_n for even n are E-cordial graphs. The same authors in [10] proved that the middle graph, total graph and split graph of P_n and the composition of P_n with P_2 admit E-cordial labeling.

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Vaidya and Vyas[11] have proved that the mirror graphs of even cycle C_n, even path P_n and hypercube Q_k are E-cordial graphs. The same authors in [12] proved that $K_n \times P_2$ and $P_n \times P_2$ are E-cordial graphs for even n while $W_n \times P_2$ and $K_{1,n} \times P_2$ are E-cordial graphs for odd n.

Definition 1.7: The *wheel graph* W_n is defined to be the join $K_1 + C_n$. The vertex corresponding to K_1 is known as apex vertex and vertices corresponding to cycle are known as *rim vertices* while the edges corresponding to cycle are known as *rim edges*. We continue to recognize apex of wheel as the apex of respective graphs obtained from wheel.

Definition 1.8: The *helm* H_n is the graph obtained from a wheel W_n by attaching a pendant edge to each rim vertex.

Definition 1.9: The *closed helm* CH_n is the graph obtained from a helm H_n by joining each pendant vertex to form a cycle.

Definition 1.10: The *flower graph* Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the apex of the helm.

Definition 1.11: The *double triangular snake* DT_n is obtained from a path P_n with vertices $v_1, v_2, \ldots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, \ldots, n-1$ and to a new vertex u_i for $i = 1, 2, \ldots, n-1$.

Definition 1.12: Let $e = uv$ be an edge of graph G and w is not a vertex of G. The edge e is subdivided when it is replaced by edges $e' = uw$ and $e'' = wv$.

Definition 1.13: The *gear graph* G_n is obtained from the wheel by subdividing each of its rim edges.

II. Main Results

Theorem 2.1: Fl_n is E - cordial.
Proof: Let H_n be a helm with v as the apex vertex, $v_1, v_2, \ldots, v_n$ be the vertices of cycle and $u_1, u_2, \ldots, u_n$ be the pendant vertices for $n > 3$. Let Fl_n be the flower graph obtained from helm H_n then $|V(Fl_n)| = 2n + 1$ and $|E(Fl_n)| = 4n$. We define $f : E(Fl_n) \to \{0, 1\}$ as follows.
For $1 \le i \le n$:

$$
\begin{aligned}
&f(vv_i) = f(v_iu_i) = 1 \\
&f(vu_i) = 0 \\
&f(v_iv_{i+1}) = 0 \qquad (v_{n+1} = v_1)
\end{aligned}
$$

In view of above defined labeling pattern f satisfies the vertex and edge conditions for E-cordial labeing as shown in *Table I*. Hence Fl_n is E-cordial graph.

Illustration 2.2: Fl_6 and its E-cordial labeling is shown in in *Figure 1*.

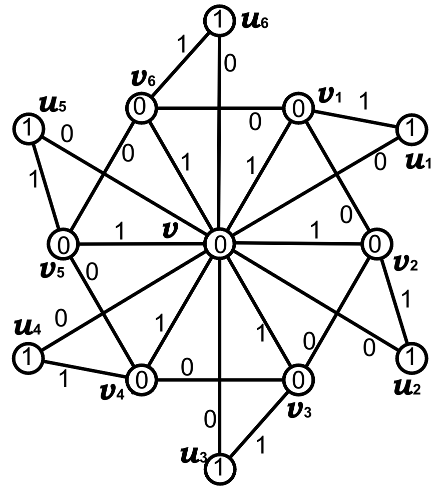

Figure 1

Theorem 2.3: CH_n is E - cordial.
Proof: Let v be the apex vertex, $v_1, v_2, \ldots, v_n$ be the vertices of inner cycle and $u_1, u_2, \ldots, u_n$ be the vertices of outer cycle of CH_n. We note that $|V(CH_n)| = 2n + 1$ and $|E(CH_n)| = 4n$. We define $f : E(CH_n) \to \{0, 1\}$ as follows:
For $1 \le i \le n$:

$$
\begin{aligned}
&f(vv_i) = 1 \\
&f(v_iu_i) = 0 \\
&f(v_iv_{i+1}) = 1 \quad (v_{n+1} = v_1) \\
&f(u_iu_{i+1}) = 0 \quad (u_{n+1} = u_1)
\end{aligned}
$$

In view of above defined labeling pattern f satisfies the vertex and edge conditions for E-cordial labeing as shown in *Table II*. Hence CH_n is E-cordial graph.

Illustration 2.4: CH_6 and its E-cordial labeling is shown in *Figure 2*.

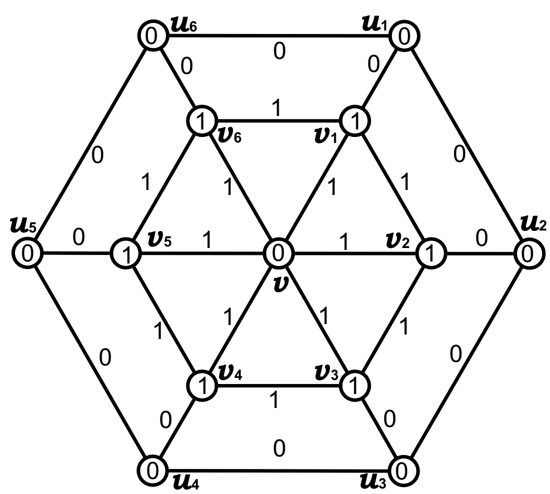

Figure 2

Theorem 2.5: DT_n is E - cordial.
Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of path P_n and $u_1, u_2, \ldots, u_n$ and $u'_1, u'_2, \ldots, u'_n$ be the newly added vertices in order to obtain DT_n. We note that $|V(DT_n)| = 3n - 2$ and $|E(DT_n)| = 5n - 5$. We define $f : E(DT_n) \to \{0, 1\}$ as follows:

Case 1: $n \equiv 1(mod\,2)$

For $1 \leq i \leq n-1$

$$f(v_iv_{i+1}) = \begin{cases} 0 & i \equiv 1,2(\text{mod}4) \\ 1 & otherwise \end{cases}$$
$$f(v_iu_i) = \begin{cases} 0 & i \equiv 1(\text{mod}2) \\ 1 & otherwise \end{cases}$$
$$f(u_iv_{i+1}) = \begin{cases} 0 & i \equiv 1(\text{mod}2) \\ 1 & otherwise \end{cases}$$
$$f(v_iu_i') = 1$$
$$f(u_i'v_{i+1}) = 0$$

Case 2: $n \equiv 0(mod\,2)$

$$f(v_1v_2) = 1$$

For $2 \leq i \leq n-1$

$$f(v_iv_{i+1}) = \begin{cases} 1 & i \equiv 0(\text{mod}2) \\ 0 & otherwise \end{cases}$$

For $1 \leq i \leq n-1$

$$f(v_iu_i) = \begin{cases} 0 & i \equiv 1(\text{mod}2) \\ 1 & otherwise \end{cases}$$
$$f(u_iv_{i+1}) = \begin{cases} 0 & i \equiv 1(\text{mod}2) \\ 1 & otherwise \end{cases}$$
$$f(v_iu_i') = 1$$
$$f(u_i'v_{i+1}) = 0$$

In view of above defined labeling pattern f satisfies the vertex and edge conditions for E-cordial labeing as shown in *Table III*. Hence DT_n is E-cordial graph.

Illustration 2.6: DT_5 and its E-cordial labeling is shown in in *Figure 3*.

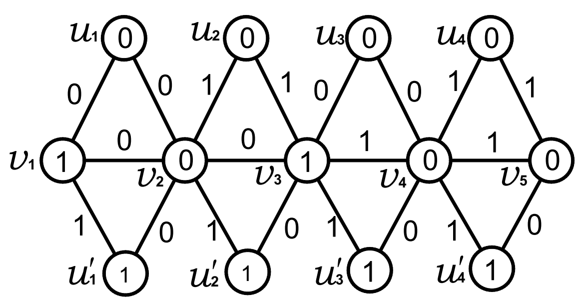

Figure 3

Theorem 2.7: G_n is E - cordial.

Proof: Let W_n be a wheel with apex vertex v and rim vertices $v_1, v_2, \ldots, v_n$. To obtain the gear graph G_n subdivide each rim edges of wheel by the vertices $u_1, u_2, \ldots, u_n$. Where each u_i is added between v_i and v_{i+1} for $i = 1, 2, \ldots, n-1$ and u_n is added between v_1 and v_n. Then $|V(G_n)| = 2n+1$ and $|E(G_n| = 3n$. We define $f : E(G_n) \rightarrow \{0, 1\}$ as follows.

Case 1: $n \equiv 1(mod\,2)$

For $1 \leq i \leq n-1$:

$$f(v_1u_1) = 1$$
$$f(u_1v_2) = 0$$
$$f(vv_i) = \begin{cases} 1 & i \equiv 1(\text{mod}2) \\ 0 & otherwise \end{cases}$$

Sub Case 1: $n \equiv 1(mod4)$
$f(vv_n) = 1$
For $2 \leq i \leq n$:

$$f(v_iu_i) = \begin{cases} 1 & i \equiv 2,3(\bmod 4) \\ 0 & otherwise \end{cases}$$
$$f(u_iv_{i+1}) = \begin{cases} 1 & i \equiv 2,3(\bmod 4) \\ 0 & otherwise \end{cases} \quad (consider\, v_{n+1} = v_1)$$

Sub Case 2: $n \equiv 3(mod4)$
$f(vv_n) = 0$
$f(v_{n-1}u_{n-1}) = 1$
$f(u_{n-1}v_n) = 1$
$f(v_nu_n) = 0$
$f(u_nv_1) = 0$
For $2 \leq i \leq n-2$:

$$f(v_iu_i) = \begin{cases} 1 & i \equiv 2,3(\text{mod}4) \\ 0 & otherwise \end{cases}$$
$$f(u_iv_{i+1}) = \begin{cases} 1 & i \equiv 2,3(\text{mod}4) \\ 0 & otherwise \end{cases}$$

Case 2: $n \equiv 0(mod\,2)$

Sub Case 1: $n \equiv 0(mod\,4)$:

$$f(vv_i) = \begin{cases} 0 & i \equiv 1(\text{mod}2) \\ 1 & otherwise \end{cases}$$
$$f(v_iu_i) = \begin{cases} 1 & i \equiv 1,2(\text{mod}4) \\ 0 & otherwise \end{cases}$$
$$f(u_iv_{i+1}) = \begin{cases} 1 & i \equiv 1,2(\text{mod}4) \\ 0 & otherwise \end{cases}$$

Sub Case 2: $n \equiv 2(mod\,4)$:
$f(vv_1) = 1$
$f(vv_n) = 0$
$f(v_nu_n) = 1$
$f(u_nv_1) = 0$
For $2 \leq i \leq n-1$:

$$f(vv_i) = \begin{cases} 1 & i \equiv 0(\text{mod}2) \\ 0 & otherwise \end{cases}$$

For $1 \leq i \leq n-1$:

$$f(v_iu_i) = \begin{cases} 1 & i \equiv 1,2(\text{mod}4) \\ 0 & otherwise \end{cases}$$
$$f(u_iv_{i+1}) = \begin{cases} 1 & i \equiv 1,2(\text{mod}4) \\ 0 & otherwise \end{cases}$$

In view of above defined labeling pattern f satisfies the vertex and edge conditions for E-cordial labeing as shown in *Table IV*. Hence G_n is E-cordial graph.

Illustration 2.8: G_7 and its E-cordial labeling is shown in *Figure 4*.

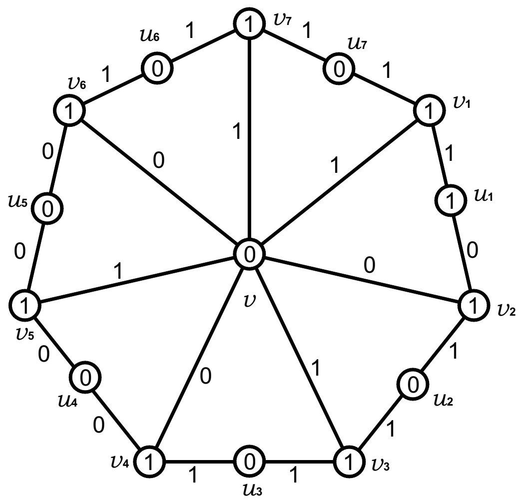

Figure 4

Concluding Remarks

Some new E-cordial graphs are investigated. To investigate some characterization(s) or sufficient condition(s) for the graph to be E-cordial is an open area of research.

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TABLE I

	Vertex Condition	*Edge Condition*
$n \equiv 0(mod\,2)$	$v_f(0) - 1 = v_f(1) = n$	$e_f(0) = e_f(1) = 2n$
$n \equiv 1(mod\,2)$	$v_f(0) = v_f(1) - 1 = n$	$e_f(0) = e_f(1) = 2n$

TABLE II

	Vertex Condition	*Edge Condition*
$n \equiv 0(mod\,2)$	$v_f(0) - 1 = v_f(1) = n$	$e_f(0) = e_f(1) = 2n$
$n \equiv 1(mod\,2)$	$v_f(0) = v_f(1) - 1 = n$	$e_f(0) = e_f(1) = 2n$

TABLE III

	Vertex Condition	*Edge Condition*
$n \equiv 0(mod\,2)$	$v_f(0) = v_f(1) = \frac{3n-2}{2}$	$e_f(0) - 1 = e_f(1) = \lfloor \frac{5n-5}{2} \rfloor$
$n \equiv 1(mod\,4)$	$v_f(0) - 1 = v_f(1) = \lfloor \frac{3n-2}{2} \rfloor$	$e_f(0) = e_f(1) = \frac{5n-5}{2}$
$n \equiv 3(mod\,4)$	$v_f(0) = v_f(1) - 1 = \lfloor \frac{3n-2}{2} \rfloor$	$e_f(0) = e_f(1) = \frac{5n-5}{2}$

TABLE IV

	Vertex Condition	*Edge Condition*
$n \equiv 1(mod\,2)$	$v_f(0) = v_f(1) - 1 = n$	$e_f(0) + 1 = e_f(1) = \frac{3n+1}{2}$
$n \equiv 0(mod\,2)$	$v_f(0) - 1 = v_f(1) = n$	$e_f(0) = e_f(1) = \frac{3n}{2}$