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CONSTRUCTION OF L-BORDERENERGETIC GRAPHS

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Abstract. If a graph G of order n has the Laplacian energy same as that of complete graph K_n then G is said to be L-borderenergeic graph. It is interesting and challenging as well to identify the graphs which are L-borderenergetic as only few graphs are known to be L-borderenergetic. In the present work we have investigated a sequence of L-borderenergetic graphs and also devise a procedure to find sequence of L-borderenergetic graphs from the known L-borderenergetic graph.

1. Introduction

Throughout this paper, we begin with finite, undirected and simple graph G. For a standard terminology and notations in graph theory we follow Balakrishnan and Ranganathan [1], while the terms related to algebra are used in the sense of Lang [8]. Throughout this paper $\overline{G}$, K_p and $\overline{K_p}$, respectively, denote complement of G, complete graph on p vertices and null graph with p vertices. The average vertex degree of G is denoted by $\overline{d}$ and defined as $\overline{d} = \frac{\sum d_i}{n}$, where d_i is degree of vertex v_i.

Let G be an undirected simple graph with vertices $v_1, v_2, \ldots, v_n$. The *adjacency matrix* denoted by $A(G)$ of G is defined to be $A(G) = [a_{ij}]$, such that, $a_{ij} = 1$ if v_i is adjacent, with v_j and 0 otherwise. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A(G)$ are known as eigenvalues of graph G. The energy $E(G)$ of graph G is defined by

$$E(G) = \sum_{i=1}^{n} |\lambda_i| .$$

The concept of graph energy was introduced by Gutman [6] in 1978. It is well known that the energy of complete graph is $2(n-1)$. In 1978 Gutman [6] conjectured that among all the graph with n vertices, the complete graph K_n has the maximum

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energy. This conjecture was disproved by Walikar et al. [12] by showing existence of graphs whose energy is greater than that of complete graphs. The graphs whose energy is $2(n-1)$ are termed as Borderenergetic according to Gong et al. [5].

Let $D(G)$ be the diagonal matrix of whose $(i,i)^{\text{th}}$ entry is the degree of a vertex v_i. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian* matrix of G. The eigenvalues of $L(G)$ are denoted by $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n$. It is well known that $L(G)$ is a positive semi definite and singular matrix. So, for $i = 1, 2, \ldots, n-1$, $\mu_i \geq 0$ and $\mu_n = 0$. The collection of all Laplacian eigenvalues together with their multiplicities is known as *Laplacian spectra* (L-spectra). Hence,

$$\operatorname{spec}_L(G) = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \mu_n = 0 \\ m(\mu_1) & m(\mu_2) & \cdots & m(\mu_{n-1}) & m(\mu_n) \end{pmatrix}.$$

The concept of Laplacian energy of G was introduced by Gutman and Zhou [7], is defined by $LE(G) = \left|\mu_i - \bar{d}\right|$, where μ_i are the Laplacian eigenvalues of G and $\bar{d}$ is the average vertex degree of G.

Recently, a concept analogous to borderenergetic graphs in the context of Laplacian energy has been introduced by Tura [10] which is teremed as L-borderenergetic graphs. According to him, a graph G of order n is said to be L-borderenergetic if $LE(G) = LE(K_n) = 2(n-1)$. Let S_n^1 be the graph obtained from an n-order star S_n by adding an edge between any two pendant vertices. Obviously, S_n^1 is an unicyclic and threshold graph. Deng et al. [3] have shown that S_n^1 is L-borderenergetic graph. Same authors [3] have established several characterizations on L-borderenergetic graphs with maximum degree at most 4.

Obviously there does not exist L-borderenergetic graph on two vertices. Hou and Tao [9] have proved that a L-borderenergetic graph on n vertices has at least n edges. As the only graph with three vertices are the paths P_3 or K_3, there does not exist a borderenergetic graphs on three vertices. By applying computer search, Hou and Tou [9] have obtained total 185 non isomorphic, non complete L-borderenergetic graphs of order upto 10. Elumalai and Rostami [4] corrected this number to 307 (see Table 1).

Table 1.

order	4	5	6	7	8	9	10
number	2	1	11	5	33	23	232

It is very interesting to investigate a graph or graph families which are L-borderenergetic because very few graphs are known to be L-borderenergetic. Here we have devised a procedure to construct a sequence of L borderenergetic graphs. We begin the next section with a definition and some existing results for the advancement of the discussion.

2. Main Result

Definition 2.1. The *join* of G_1 and G_2 is a graph $G = G_1 \vee G_2$ with vertex set $V(G_1) \cup V(G_2)$ and an edge set consisting of all the edges of G_1 and G_2 together with the edges joining each vertex of G_1 with every vertex of G_2.

Proposition 2.1 ([2]). *Let G_1 and G_2 be graphs of n_1 and n_2 vertices, respectively. If $\alpha_1, \alpha_2, \dots, \alpha_{n_1-1}, \alpha_{n_1} = 0$ and $\beta_1, \beta_2, \dots, \beta_{n_2-1}, \beta_{n_2} = 0$ be L-spectra of G_1 and G_2, respectively. Then the L-spectra of $G_1 \vee G_2$ are*

$$n_2 + \alpha_1, n_2 + \alpha_2, \dots, n_2 + \alpha_{n_1-1}, n_1 + \beta_1, n_1 + \beta_2, \dots, n_1 + \beta_{n_2-1}, n_1 + n_2, 0.$$

Theorem 2.1. *Let G be a L-borderenergetic graph of order n with average vertex degree $\bar{d} \in \mathbb{Z}$. Then for $p \neq 0$, $G \vee \overline{K_p}$ is L-borderenergetic if $p = n - \bar{d}$.*

Proof. Let $\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n = 0$ be L-spectra of G. As G is L-borderenergetic of order n, $LE(G) = 2n - 2$, which implies that

$$\sum_{i=1}^{n} \left|\mu_i - \bar{d}\right| = 2n - 2.$$

Hence,

$$\sum_{i=1}^{n-1} \left|\mu_i - \bar{d}\right| = 2n - 2 - \bar{d}. \tag{2.1}$$

By Proposition 2.1, L-spectra of $G \vee \overline{K_p}$ is

$$\operatorname{spec}_L(G) = \begin{pmatrix} \mu_1 + p & \mu_2 + p & \cdots & \mu_{n-1} + p & n & n + p & 0 \\ 1 & 1 & \cdots & 1 & p - 1 & 1 & 1 \end{pmatrix}.$$

If $\bar{d}'$ is average vertex degree of newly constructed graph $G \vee \overline{K_p}$, then

$$\bar{d}' = \frac{n\bar{d} + 2np}{n + p}.$$

Note that for each $1 \leq i \leq n - 1$

$$\begin{aligned} \mu_i + p - \bar{d}' &= \mu_i + p - \frac{n\bar{d} + 2np}{p + n} \\ &= \mu_i - \bar{d} + \left(p + \bar{d} - \frac{n\bar{d} + 2np}{p + n}\right) \\ &= \mu_i - \bar{d} - \frac{p(n - p - \bar{d})}{p + n}. \end{aligned}$$

Now,

$$LE(G \vee \overline{K_p}) = \sum_{i=1}^{n-1} \left|\mu_i + p - \bar{d}'\right| + (p - 1)\left|n - \bar{d}'\right| + \left|n + p - \bar{d}'\right| + \left|\bar{d}'\right|$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-1} \left| \mu_i - \bar{d} - \frac{p(n-p-\bar{d})}{p+n} \right| + (p-1) \left| n - \frac{n\bar{d}+2np}{n+p} \right| \\
&\quad + \left| n + p - \frac{n\bar{d}+2np}{n+p} \right| + \left| \frac{n\bar{d}+2np}{n+p} \right| \\
&= \sum_{i=1}^{n-1} \left| \mu_i - \bar{d} - \frac{p(n-p-\bar{d})}{p+n} \right| + (p-1) \left| \frac{n(n-p-\bar{d})}{n+p} \right| \\
&\quad + \left| p + \frac{n(n-p-\bar{d})}{n+p} \right| + \left| n - \frac{n(n-p-\bar{d})}{n+p} \right|.
\end{aligned}
$$

If $p = n - \bar{d}$, then

$$LE(G \vee \overline{K_p}) = \sum_{i=1}^{n-1} \left| \mu_i - \bar{d} \right| + |p| + |n|.$$

Therefore, by (2.1), $LE(G \vee \overline{K_p}) = 2n - 2 - \bar{d} + p + n = 2n + 2p - 2 = 2(n + p - 1)$. Hence, $G \vee \overline{K_p}$ is L-borderenergetic. □

3. Sequence of L-Borderenergetic Graphs

In this section we construct an infinite sequence of L-borderenergetic graphs. We term the graph under consideration as underlying graph. To construct the sequence we take any L-borderenergetic graphs of order n with average vertex degree $\bar{d} \in \mathbb{Z}$ as underlying graph and then the sequence is obtained by joining $n - \bar{d}$ vertices at each iteration.

Let $G^{(0)}$ is any L-borderenergetic graph of order n with average vertex degree $\bar{d} \in \mathbb{Z}$. Consider an infinite sequence of graphs $\mathcal{H} = \{G^{(0)}, G^{(1)}, \dots, G^{(k)}, \dots\}$ such that

$$G^{(1)} = G^{(0)} \vee \overline{K_{n-\bar{d}}},\ G^{(2)} = G^{(1)} \vee \overline{K_{n-\bar{d}}}, \dots, G^{(k)} = G^{(k-1)} \vee \overline{K_{n-\bar{d}}}, \dots$$

Note that each $G^{(k)}$ is of order $n + k(n - \bar{d})$ with average vertex degree $d_k = \bar{d} + k(n - \bar{d})$.

Lemma 3.1. *Let $G^{(0)}$ be a graph of order n with average vertex degree $\bar{d} \in \mathbb{Z}$ with Laplacian eigenvalues $\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n = 0$. Then for any $G^{(k)} \in \mathcal{H}$, $k \geq 1$, the Laplacian spectrum of $G^{(k)}$ is*

$$
\begin{aligned}
&\operatorname{spec}_L(G^{(k)}) \\
&= \begin{pmatrix} \mu_1 + k(n-\bar{d}) & \cdots & \mu_{n-1} + k(n-\bar{d}) & n + (k-1)(n-\bar{d}) & n + k(n-\bar{d}) & 0 \\ 1 & \cdots & 1 & k(n-\bar{d}-1) & k & 1 \end{pmatrix}.
\end{aligned}
$$

Proof. We prove this result by taking induction on k. From Theorem 2.1, it is clear that result is true for $k = 1$. Assume that the result is true for $k = s - 1$. Then by induction hypothesis

$$
\begin{aligned}
&\operatorname{spec}_L(G^{(s-1)}) \\
&= \begin{pmatrix} \mu_1 + (s-1)(n-\bar{d}) & \cdots & \mu_{n-1} + (s-1)(n-\bar{d}) & n + (s-2)(n-\bar{d}) & n + (s-1)(n-\bar{d}) & 0 \\ 1 & \cdots & 1 & (s-1)(n-\bar{d}-1) & (s-1) & 1 \end{pmatrix}.
\end{aligned}
$$

For $k = s$, $G^{(s)} = G^{(s-1)} \vee \overline{K_{n-\bar{d}}}$, from Proposition 2.1,

$$\operatorname{spec}_L(G^{(s)}) = \begin{pmatrix} \mu_1 + s(n-\bar{d}) & \cdots & \mu_{n-1} + s(n-\bar{d}) & n + (s-1)(n-\bar{d}) & n + s(n-\bar{d}) & 0 \\ 1 & \cdots & 1 & s(n-\bar{d}-1) & s & 1 \end{pmatrix}.$$

Thus, the result is true for all $s \in \mathbb{N}$. Hence, by induction the result follows. □

Theorem 3.1. *For each $r \geq 1$, $G^{(k)} \in \mathcal{H}$ is L-borderenergetic with $K_{n+k(n-\bar{d})}$ for each $k \geq 1$.*

Proof. We have already shown that the order and average vertex degree of $G^{(k)}$ are $n + k(n-\bar{d})$ and $d_k = \bar{d} + k(n-\bar{d})$, respectively, for each $k \geq 1$.

$$\begin{aligned} LE(G^{(k)}) = & \sum_{i=1}^{n-1} \left| \mu_i + k(n-\bar{d}) - \bar{d} - k(n-\bar{d}) \right| \\ & + k(n-\bar{d}-1) \left| n + (k-1)(n-\bar{d}) - \bar{d} - k(n-\bar{d}) \right| \\ & + k \left| n + k(n-\bar{d}) - \bar{d} - k(n-\bar{d}) \right| + \left| \bar{d} + k(n-\bar{d}) \right| \\ = & \sum_{i=1}^{n-1} \left| \mu_i - \bar{d} \right| + k \left(n - \bar{d} \right) + \bar{d} + k(n-\bar{d}) \\ = & 2n - 2 - \bar{d} + 2k(n-\bar{d}) + \bar{d} \\ = & 2(n + k(n-\bar{d}) - 1) = LE(K_{n+k(n-\bar{d})}). \end{aligned}$$

Hence, $G^{(k)}$ is L-borderenergetic with $K_{n+k(n-\bar{d})}$ for each $k \geq 1$. □

4. Some More Sequences From Known L-Borderenergetic Graphs

In this section we construct two infinite sequences of L-borderenergetic graphs $\mathcal{G}_i = \{G_i^{(0)}, G_i^{(1)}, \ldots, G_i^{(k)}, \ldots\} \subseteq \mathcal{H}$ for $i = 1, 2$, by taking some known L-borderenergetic graphs as underlying graph.

4.1. **The sequence of S_n^1.** Let $G_1^{(0)} = S_n^1$ be the graph obtained form n-order star S_n by adding a single edge. Note that S_n^1 is a graph of order n with average degree 2,

$$\operatorname{spec}_L(S_n^1) = \begin{pmatrix} 0 & 1 & 3 & n \\ 1 & n-3 & 1 & 1 \end{pmatrix}, \quad LE(G_1^{(0)}) = 2(n-1),$$

and thus it is L-borderenergetic with K_n. Consider an infinite sequence or borderenergetic graphs $\mathcal{G}_1 = \{G_1^{(0)}, G_1^{(1)}, G_1^{(2)}, \ldots, G_1^{(k)}, \ldots\}$ such that

$$G_1^{(1)} = G_1^{(0)} \vee \overline{K_{n-2}},\ G_1^{(2)} = G_1^{(1)} \vee \overline{K_{n-2}},\ G_1^{(3)} = G_1^{(2)} \vee \overline{K_{n-2}}, \ldots$$

The parameters $n, \bar{d}, LE$ of the sequence of S_n^1 are depicted in following Table 2.

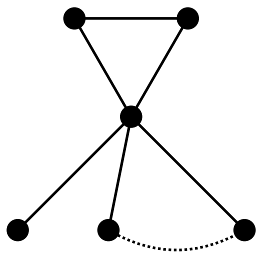

FIGURE 1. The graph S_n^1

TABLE 2.

G	n	$\overline{d}$	*L-spectra*	$LE(G)$	*L-Borderenergetic With*
$G_1^{(0)}$	n	2	$0^1, 1^{(n-3)}, 3^1, n^1$	$2(n-1)$	K_n
$G_1^{(1)} = G_1^{(0)} \vee \overline{K_{n-2}}$	$2n-2$	n	$0^1, n^{(n-3)}, (n-1)^{(n-3)}, (n+1)^1, (2n-2)^2$	$2(2n-3)$	K_{2n-2}
$G_1^{(2)} = G_1^{(1)} \vee \overline{K_{n-2}}$	$3n-4$	$2n-2$	$0^1, (2n-2)^{(2n-6)}, (2n-3)^{(n-3)}, (2n-1)^1, (3n-4)^3$	$2(3n-5)$	K_{3n-3}
$G_1^{(3)} = G_1^{(2)} \vee \overline{K_{n-2}}$	$4n-6$	$3n-4$	$0^1, (3n-4)^{(3n-9)}, (3n-5)^{(n-3)}, (3n-3)^1, (4n-6)^4$	$2(4n-7)$	K_{4n-4}
$G_1^{(4)} = G_1^{(3)} \vee \overline{K_{n-2}}$	$5n-8$	$4n-6$	$0^1, (4n-6)^{(4n-12)}, (4n-7)^{(n-3)}, (4n-5)^1, (5n-8)^5$	$2(5n-9)$	K_{5n-5}
$G_1^{(5)} = G_1^{(4)} \vee \overline{K_{n-2}}$	$6n-10$	$5n-8$	$0^1, (4n-6)^{(5n-15)}, (4n-7)^{(n-3)}, (4n-5)^1, (5n-8)^6$	$2(6n-11)$	K_{6n-6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

4.2. **The sequence of $K_{n-1} \odot K_n$.** For each integer $n \geq 3$, the graph $K_{n-1} \odot K_n$ is defined by

$$G = (K_{n-1} \cup K_{n-2}) \vee K_2.$$

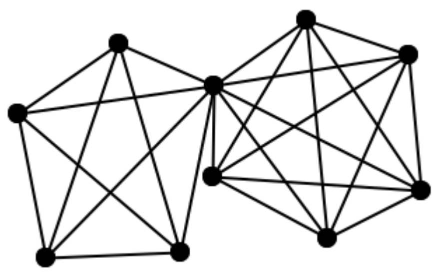

FIGURE 2. The graph $K_5 \odot K_6$

Tura [11] has proved that the $K_{n-1} \odot K_n$ is a graph with avrgare vertex degree $n-1$ and it is borderenergetic with K_{2n-2},

$$\operatorname{spec}_L(K_{n-1} \odot K_n) = \begin{pmatrix} 0 & 1 & n-1 & n & 2n-2 \\ 1 & 1 & n-3 & n-2 & 1 \end{pmatrix}, \quad LE(K_{n-1} \odot K_n) = 2(2n-3).$$

Consider an infinite sequence or borderenergetic graphs

$$\mathcal{G}_2 = \{G_2^{(0)}, G_2^{(1)}, G_2^{(2)}, \ldots, G_2^{(k)}, \ldots\},$$

such that

$$G_2^{(1)} = G_2^{(0)} \vee \overline{K_{n-1}}, \quad G_2^{(2)} = G_2^{(1)} \vee \overline{K_{n-1}}, \quad G_2^{(3)} = G_2^{(2)} \vee \overline{K_{n-1}}, \dots$$

The parameters n, $\bar{d}$, LE of the sequence of borderenergetic graphs are depicted in following Table 3.

Table 3.

G	**n**	$\bar{d}$	L-***spectra***	***LE(G)***	L-***Borderenergetic With***
$G_2^{(0)}$	$2n-2$	$n-1$	$0^1, 1^1, (n-1)^{(n-3)}, n^{(n-2)}, (2n-2)^1$	$2(2n-3)$	K_{2n-2}
$G_2^{(1)} = G_2^{(0)} \vee \overline{K_{n-1}}$	$3n-3$	$2n-2$	$0^1, n^1, (2n-2)^{(2n-5)}, (2n-1)^{(n-2)}, (3n-3)^2$	$2(3n-4)$	K_{3n-3}
$G_2^{(2)} = G_2^{(1)} \vee \overline{K_{n-1}}$	$4n-4$	$3n-3$	$0^1, (2n-1)^1, (3n-3)^{(3n-7)}, (3n-2)^{(n-2)}, (4n-4)^3$	$2(4n-5)$	K_{4n-4}
$G_2^{(3)} = G_2^{(2)} \vee \overline{K_{n-1}}$	$5n-5$	$4n-4$	$0^1, (3n-2)^1, (4n-4)^{(4n-9)}, (4n-3)^{(n-2)}, (5n-5)^4$	$2(5n-6)$	K_{5n-5}
$G_2^{(4)} = G_2^{(3)} \vee \overline{K_{n-1}}$	$6n-6$	$5n-5$	$0^1, (4n-3)^1, (5n-5)^{(5n-11)}, (5n-4)^{(n-2)}, (6n-6)^5$	$2(6n-7)$	K_{6n-6}
$G_2^{(5)} = G_2^{(4)} \vee \overline{K_{n-1}}$	$7n-7$	$6n-6$	$0^1, (5n-4)^1, (6n-6)^{(6n-13)}, (6n-5)^{(n-2)}, (7n-7)^6$	$2(7n-8)$	K_{7n-7}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

5. Concluding Remarks

Here we have explored the concept of L-borderenergetic graphs which is analogous to the concept of borderenergetic graphs. We have investigated a sequence of L-borderenergetic graphs in the scenario when only handful graphs are known to be L-borderenergetic. The derived result is used for the construction of two sequences of L-borderenergetic graphs from the known L-borderenergetic graphs.

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