Congruent Domination Number of Graphs Obtained by Means of Some Graph Operation

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Abstract— **A dominating set $D \subseteq V(G)$ is said to be a congruent dominating set of a graph G if $\sum_{v\in V(G)} d(v) \equiv 0(\text{mod } \sum_{v\in D} d(v))$. The minimum cardinality of a minimal congruent dominating set of G is called the congruent domination number of G which is denoted by $\gamma_{cd}(G)$. In this paper, we investigate congruent domination number of some graphs obtained by means of some graph operation.**

Keywords— **Dominating Set, Domination Number, Congruent Dominating Set, Congruent Domination Number**

I. Introduction

Domination in graphs is one of the concepts in graph theory that has piqued the interest of many researchers due to its potential to solve real-world problems involving communication network design and analysis, as well as defence surveillance. There are numerous domination models available in the literature. [1, 4, 6, 7, 8, 9] provide a concise explanation of dominating sets and related concepts. For standard notations and graph theoretic terminology, we follow West [17] while the terms related to number theory are used in the sense of Burton [2].

We begin with finite, undirected and simple graph $G = (V(G), E(G))$ of order n. A set $D \subseteq V(G)$ of vertices in a graph G is called a dominating set if each vertex in $V(G) - D$ is adjacent to at least one vertex of D. A dominating set D is a minimal dominating set if no proper subset D' of D is a dominating set of graph G. The domination number $\gamma(G)$ is the minimum cardinality of a minimal dominating set.

The following new concept is recently introduced and further explored by Vaidya and Vadhel [13, 14, 15, 16].

A dominating set $D \subseteq V(G)$ is said to be a congruent dominating set of G if

$$\sum_{v\in V(G)} d(v) \equiv 0(\text{mod } \sum_{v\in D} d(v)) \tag{1}$$

A congruent dominating set $D \subseteq V(G)$ is said to be a minimal congruent dominating set if no proper subset D' of D is congruent dominating set. The minimum cardinality of a minimal congruent dominating set of G is called the congruent domination number of G which is denoted by $\gamma_{cd}(G)$.

In the present paper we have investigated the congruent domination number of some graph obtained by means of some graph operation like Corona product, square graph of a graph, complement graph of a graph and extended double cover of a graph. The domination number of the Cartesian product of paths and cycles have been investigated in [5, 10, 11]. We have also investigated the exact value of congruent domination number for Cartesian product of cycles and paths.

The complement $\bar{G}$ of a graph G is the graph with vertex set $V(G)$ and two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

The square graph G^2 of a graph G with vertex set $V(G)$ is the graph obtained by joining every pair of vertices which are at distance two in G.

The corona $G \circ H$ of two graphs G and H (with order n and m respectively) is defined as a graph obtained by taking one copy of G and n copies of H and joining the i^{th} vertex of G with an edge to every vertex in the i^{th} copy of H.

The Cartesian product of two graphs $G(V_1, E_1)$ and $H(V_2, E_2)$, denoted by $G \square H$, is the graph with vertex set is $V_1 \times V_2$ and edge set $E(G \square H) = \{((g_1, h_1), (g_2, h_2)) : g_1 = g_2 \text{ and } (h_1, h_2) \in E_2 \text{ or } h_1 = h_2 \text{ and } (g_1, g_2) \in E_1\}$.

Definition 1.1 *The extended double cover of a graph G with the vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ is a bipartite graph G' with bipartition (X, Y) ; $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$, where two vertices x_i and y_j are adjacent if and only if $i = j$ or v_i is adjacent to v_j in G.*

We state the following results as our ready references:

Theorem 1.2 *[11] For $n \geqslant 3$,* $\gamma(C_n \square P_2) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 1 & ; \text{if } n \equiv 2 (\text{mod } 4) \\ \left\lceil \frac{n}{2} \right\rceil & ; \text{otherwise.} \end{cases}$

Theorem 1.3 *[3] Let G be a connected graph of order m and let H be any graph of order n. Then $\gamma(G \circ H) = m$.*

II. MAIN RESULTS

Theorem 2.1 *Let $m \equiv 0(mod\ \ n)$, then $\gamma_{cd}(G \circ K_1) = \gamma_{cd}(G \circ K_2) = n$.*

Proof. Let G be a graph with n vertices and m edges.
Let $m \equiv 0(\text{mod } n)$, then $2m \equiv 0(\text{mod } n)$ and $2m \equiv 0(\text{mod } 2n)$.
This implies that $\sum_{v\in V(G)} d(v) \equiv 0(\text{mod } n)$ and $\sum_{v\in V(G)} d(v) \equiv 0(\text{mod } 2n)$ (2)
Now, consider the graphs $G \circ K_1$ with $V(G \circ K_1) = \{v_1, v_2, \dots, v_n, v_{1\prime}, v_{2\prime}, \dots, v_{n\prime}\}$, where $d(v_i) = d_G(v_i) + 1$, $d(v_{i\prime}) = 1$, for $1 \leqslant i \leqslant n$ and $G \circ K_2$ with $V(G \circ K_2) = \{v_1, v_2, \dots, v_n, v_1', v_2', \dots, v_n', v_1'', v_2'', \dots, v_n''\}$, where $d(v_i) = d_G(v_i) + 1$, $d(v_i') = 2$ and $d(v_i'') = 1$, for $1 \leqslant i \leqslant n$.
Moreover, $\sum_{v\in V(G\circ K_1)} d(v) = \sum_{v\in V(G)} d(v) + 2n$ and $\sum_{v\in V(G\circ K_2)} d(v) = \sum_{v\in V(G)} d(v) + 4n$ (3)
Let $D = \{v_1', v_2', \dots, v_n'\}$, then D is a dominating set for both $G \circ K_1$ and $G \circ K_2$ with $|D| = n$. Moreover, D is a minimal domination set with minimum cardinality as $\gamma(G \circ K_1) = \gamma(G \circ K_2) = n$.
Moreover, $\sum_{v\in D(G\circ K_1)} d(v) = \sum_{i=1}^{n} d_{G\circ K_1}(v_i') = n$ and $\sum_{v\in D(G\circ K_2)} d(v) = \sum_{i=1}^{n} d_{G\circ K_2}(v_i') = 2n$ (4)
Thus, from (1), (2) and (3), we get, D is a minimal congruent dominating set with minimum cardinality.
Hence, $\gamma_{cd}(G \circ K_1) = \gamma_{cd}(G \circ K_2) = n$.

Theorem 2.2 *Let G be an r-regular graph, then $\gamma_{cd}(G \circ K_1) = n$.*

Proof. Let G be an r-regular graph with $V(G) = \{v_1, v_2, \dots, v_n\}$.
Now, consider the graph $G \circ K_1$ with $V(G \circ K_1) = \{v_1, v_2, \dots, v_n, v_1', v_2', \dots, v_n'\}$, where for each i, v_i' is the corresponding vertex of v_i. Then

$$\sum_{v\in V(G)} d(v) = \sum_{i=1}^{n} d(v_i) + \sum_{i=1}^{n} d(v_i') = n(r + 1) + n = n(r + 2) \quad (5)$$

Let $D = \{v_1', v_2', \dots, v_n'\}$, then D is a minimal domination set with minimum cardinality as $\gamma(G \circ K_1) = n$.
Moreover, $\sum_{v\in D} d(v) = \sum_{i=1}^{n} d(v_i') = n$ (6)
Thus, from (5) and (6), D satisfies Condition (1) to be a congruent dominating set. Hence, $\gamma_{cd}(G \circ K_1) = n$.

Theorem 2.3 *For the square of path P_n,*

$$\gamma_{cd}(P_n^2) = \begin{cases} \frac{n}{3} & ; \text{if } n \equiv 0(\text{mod}3) \\ \frac{n-1}{2} & ; \text{if } n \text{ is prime} \\ \frac{n}{2} & ; \text{Otherwise.} \end{cases}$$

Proof. Let $V = \{v_1, v_2, \dots, v_n\}$ be the set of vertices of path P_n^2 with $|V| = n$.
Here, v_1 and v_n are the vertices of degree 2, v_2 and v_{n-1} are the vertices of degree 3 and $v_3, v_4, \dots, v_{n-2}$ are the vertices of degree 4.
Therefore, $\sum_{v\in V(G)} d(v) = 2(2) + 2(3) + 4(n - 4) = 4n - 6 = 2(2n - 3)$.

Case-1: $n \equiv 0(\text{mod}3)$
Define $D = \{v_{3k+1} / 0 \leqslant k \leqslant \frac{n}{3} - 1\}$ with $|D| = \frac{n}{3}$. Then D is a minimal dominating set as $D - \{v_i\}$ doesn't dominate vertex v_i, for each $0 \leqslant i \leqslant n$.
Moreover, there is 1 vertex of degree 2 and all other vertices of degree 4 and so $\sum_{v\in D} d(v) = 2 + 4\left(\frac{n}{3} - 1\right) = \frac{2}{3}(2n - 3)$.

Then, D satisfies Condition (1) to be a congruent dominating set. Therefore, D is a congruent dominating set.
Since, D is a minimal dominating set, it is a minimal congruent dominating set.
Let $S \subset V(G)$ be the set of vertices with $|S| < |D|$ and $\sum_{v\in S} d(v) = t < \frac{2}{3}(2n-3)$.
Then, there does not exist any $t \in \mathbb{N}$ with $t < \frac{2}{3}(2n-3)$ such that $t < 2n-1$ and $t|2(2n-1)$.
Hence, D is a minimal congruent dominating set with minimum cardinality.

Case 2: n is prime.
Define $D = \left\{v_2 \cup v_{2k+1}/0 \leqslant k \leqslant \frac{n-3}{2}\right\}$ with $|D| = \frac{n-1}{2}$. Then D is a dominating set.
Moreover, there is one vertex of degree 3 and all other vertex of degree 4 in set D and so, $\sum_{v\in D} d(v) = 3 + 4\left(\frac{n-3}{2}\right) = (2n - 3)$. Then, D satisfies Condition (1) to be a congruent dominating set. Therefore, D is a congruent dominating set.
Moreover, D is a minimal congruent dominating set as $D - \{v\}$; $\forall v \in V(G)$ is not a congruent dominating set.
Let $S \subset V(G)$ be the set of vertices with $\sum_{v\in S} d(v) = t < (2n-3)$.
Then, there does not exist any $t \in \mathbb{N}$ such that $t < 2n-3$ and $t|2(2n-1)$.
Hence, D is a minimal congruent dominating set with minimum cardinality.
Case-3: Define $D = \left\{v_{2k+1}/0 \leqslant k \leqslant \frac{n}{2} - 1\right\}$ with $|D| = \frac{n}{2}$. Then D is a dominating set.
Moreover, there is one vertex of degree 2, one vertex of degree 3 and $\left(\frac{n}{2} - 2\right)$ vertices of degree 4 in D and so,
$\sum_{v\in D} d(v) = 2 + 3 + 4\left(\frac{n}{2} - 2\right) = (2n-3)$.
Then, $\sum_{v\in V(G)} d(v) \equiv 0(\text{mod } \sum_{v\in D} d(v))$. Therefore, D is a congruent dominating set.
Moreover, D is a minimal congruent dominating set as $D - \{v\}; \forall v \in V(G)$ is not a congruent dominating set.
Let $S \subset V(G)$ be the set of vertices with $\sum_{v\in S} d(v) = t < (2n-3)$.
Then, there does not exist any $t \in \mathbb{N}$ such that $t < 2n-3$ and $t|2(2n-1)$.
This implies that D is a minimal congruent dominating set with minimum cardinality.
Hence,

$$\gamma_{cd}(G) = \begin{cases} \frac{n}{3} & ; \text{if } n \equiv 0(\text{mod}3) \\ \frac{n-1}{2} & ; \text{if } n \text{ is prime} \\ \frac{n}{2} & ; \text{Otherwise.} \end{cases}$$

Theorem 2.4 *For the book graph B_n,*

$$\gamma_{cd}(B_n) = \begin{cases} n+1 & ; \text{if } n = 1 \text{ or } n \text{ is even} \\ \frac{n+3}{2} & ; \text{Otherwise} \end{cases}$$

Proof. Let $V(G) = \{u, v, u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ be the set of vertices of B_n, where u and v are the apex vertices of degree $n+1$ and all other vertices are of degree 2.
Then, $\sum_{v\in V(G)} d(v) = 2(n+1) + 2(2n) = 6n + 2 = 2(3n+1)$.

Case-1: $n = 1$ or n is even.
Consider a set $D \subseteq V(G)$ as follows:
$D = \{u, v_1, v_2, \dots, v_n\}$ with $|D| = n+1$.
Then, D is a minimal dominating set as for any $x \in D$, $D - \{x\}$ will not dominate x.
Here, $\sum_{v\in D} d(v) = (n+1) + 2(n) = 3n + 1$.
Therefore, D satisfies Condition (1) to be a congruent dominating set. Hence, D is a congruent dominating set.
Moreover, D is a minimal congruent dominating set as it is a minimal dominating set.
Select $i \in \mathbb{N}$ such that $i|6n+2$.
Now consider the set of vertices S of $V(G)$ such that $|S| < |D|$ with $\sum_{v\in S} d(v) = \frac{6n+2}{i}$.
Then, $\left(\frac{6n+2}{i}\right)|6n+2$.
But for each $i > 2$, $|S| < 2$ and $\gamma(B_n) = 2$.
Therefore, no such congruent dominating set S exists such that $|S| < n+1$.
This implies that D is a minimal congruent dominating set with minimum cardinality.

Case-2: $n > 1$ is odd.
Consider a set $D \subseteq V(G)$ as follows:
$D = \left\{u, v, u_{2k} / 0 \leqslant k \leqslant \frac{n-3}{2}\right\}$ with $|D| = \frac{n+3}{2}$. Then D is a dominating set.
Here, $\sum_{v \in D} d(v) = 2(n+1) + 2\left(\frac{n-1}{2}\right) = 3n + 1$. Therefore, D satisfies Condition (1) to be a congruent dominating set. Hence, D is a congruent dominating set.
Select $i \in \mathbb{N}$ such that $i | 6n + 2$.
Now consider the set of vertices S of $V(G)$ such that $|S| < |D|$ with $\sum_{v \in S} d(v) = \frac{6n+2}{i}$.
Then, $\left(\frac{6n+2}{i}\right) | 6n + 2$.
But for each $i > 2$, $|S| < 2$ and $\gamma(B_n) = 2$.
Therefore, no such congruent dominating set S exists such that $|S| < \frac{n+3}{2}$.
This implies that D is a minimal congruent dominating set with minimum cardinality.
Hence, $\gamma_{cd}(B_n) = \begin{cases} n+1 & ; \text{if } n = 1 \text{ or } n \text{ is even} \\ \frac{n+3}{2} & ; \text{otherwise.} \end{cases}$

Theorem 2.5 *Let G be the complement of path P_n with $n > 2$, then*
$$\gamma_{cd}(G) = \begin{cases} 2 & ; \text{if } n \text{ is odd} \\ \frac{n}{2} & ; \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $V(G) = \{v_1, v_2, \ldots\ldots, v_n\}$ be the set of vertices of graph G.
Then, $d(v_1) = d(v_n) = n - 2$ and $d(v_i) = n - 3$; $\forall\ 2 \leqslant i \leqslant n - 1$ and so,
$\sum_{v \in V(G)} d(v) = 2(n-2) + (n-2)(n-3) = (n-1)(n-2)$.

Case-1: n is odd.
Consider a subset $D \subseteq V(G)$ of vertices as follows:
$D = \{v_1, v_2\}$ with $|D| = 2$. Then D is a minimal dominating set with minimum cardinality as $\gamma_{cd}(\overline{P_n}) = 2$.
Now, $\sum_{v \in D} d(v) = 2(n-2)$ and $\sum_{v \in V(G)} d(v) = (n-1)(n-2)$.
Here, $2(n-2) | (n-1)(n-2)$ this implies that $2 | n - 1$ as n is odd and so, D satisfies Condition (1) to be a congruent dominating set.
Since D is a minimal dominating set with minimum cardinality it is also a minimal congruent dominating set with minimum cardinality.

Case-2: n is even.
Consider a subset $D \subseteq V(G)$ of vertices as follows:
$D = \left\{v_1, v_2, \cdots, v_k / 1 \leqslant k \leqslant \frac{n}{2}\right\}$ with $|D| = \frac{n}{2}$. Then D is a dominating set.
Now, $\sum_{v \in D} d(v) = (n-2) + \left(\frac{n}{2} - 1\right)(n-3) = \frac{(n-1)(n-2)}{2}$.
Then, D satisfies Condition (1) to be a congruent dominating set. Therefore, D is a congruent dominating set.
Now $\gamma(\overline{P_n}) = 2$ and so $2 \leqslant \gamma_{cd}(\overline{P_n})$.
Select $i \in \mathbb{N}$ such that $i | (n-1)(n-2)$.
Now consider the set of vertices S of $V(G)$ such that $|S| < |D|$ with $\sum_{v \in S} d(v) = \frac{(n-1)(n-2)}{i}$.
Then, $\left(\frac{(n-1)(n-2)}{i}\right) | (n-1)(n-2)$.
But for each $i > 2$, $|S| < 2$ and $\gamma(\overline{P_n}) = 2$.
Therefore, no such congruent dominating set S exists such that $|S| < \frac{n}{2}$.
This implies that D is a minimal congruent dominating set with minimum cardinality.
Hence, $\gamma_{cd}(G) = \begin{cases} 2 & ; \text{if } n \text{ is odd} \\ \frac{n}{2} & ; \text{if } n \text{ is even.} \end{cases}$

Theorem 2.6 *Let G be the extended double cover of cycle C_n, then*

$$\gamma_{cd}(G) = \begin{cases} \frac{n}{2} & ;\text{if } n \equiv 0(\text{mod } 4) \\ \frac{2n}{3} & ;\text{if } n \equiv 0(\text{mod } 3)\ \&\ n \not\equiv 0(\text{mod } 4) \\ n & ;\text{otherwise.} \end{cases}$$

Proof. Let $v_1, v_2, \dots v_n$ be the vertices of cycle C_n. Then, $V(G) = \{v_1, v_2, \dots v_n, v_1', v_2', \dots, v_n'\}$ is the set of vertices of graph G, where G is the extended double cover of cycle C_n and $d(v) = 3,\ \forall v \in V(G)$. Moreover, $\sum_{v\in V(G)} d(v) = 6n$.

Case-1: $n \equiv 0(\text{mod } 4)$
Consider $D = \left\{v_{4k+1} \cup v_{4k+3}'/0 \leqslant k \leqslant \frac{n}{4} - 1\right\}$, then $|D| = \frac{n}{2}$. Moreover, D is a minimal dominating set with minimum cardinality with degree sum of vertex set of dominating set D is $\frac{3n}{2}$. Hence, D satisfies the condition (1) for being a congruent dominating set.
Since D is a minimal dominating set with minimum cardinality, it is also a minimal congruent dominating set with minimum cardinality.
Thus, $\gamma_{cd}(G) = \frac{n}{2}$, in this case.

Case-2: $n \equiv 0(\text{mod } 3)\ \&\ n \not\equiv 0(\text{mod } 4)$
Consider $D = \left\{v_{3k+1} \cup v_{3k+2}'/0 \leqslant k \leqslant \frac{n}{3} - 1\right\}$, then $|D| = \frac{2n}{3}$ and D is a dominating set. Moreover, the degree sum of vertex set of a dominating set D is $2n$. Hence, D satisfies the condition (1) for being a congruent dominating set.
We claim that D is of minimal cardinality.
Let $i \in \mathbb{N}$ such that $i|6n$.
If possible, let $S \neq D,\ S \subset V(G)$ with $|S| < |D|$ and degree sum of vertices of S is $\frac{6n}{i}$.
Then, $\left(\frac{6n}{i}\right)|6n$.
But for each $i > 3$, $|S| < 2\left\lceil\frac{n}{3}\right\rceil$ and $\gamma(G) = 2\left\lceil\frac{n}{3}\right\rceil$.
Therefore, no such congruent dominating set S exists such that $|S| < n$.
This implies that D is a minimal congruent dominating set with minimum cardinality.
Thus, $\gamma_{cd}(G) = \frac{2n}{3}$, in this case.

Case-3: $n \not\equiv 0(\text{mod } 3{,}4)$
Consider $D = \{v_1, v_2, \dots, v_n\}$, then $|D| = n$ and D is a dominating set with degree sum of vertex set of dominating set D is $3n$. Hence, D satisfies the condition (1) for being a congruent dominating set.
Since, $n \not\equiv 0(\text{mod } 3{,}4)$ and $\forall v \in V(G),\ d(v) = 3$ implies there does not exist any congruent dominating set S with $|S| < |D|$ and so, D is a minimal congruent dominating set with minimum cardinality.
Thus, $\gamma_{cd}(G) = n$, in this case.
Hence,

$$\gamma_{cd}(G) = \begin{cases} \frac{n}{2} & ;\text{if } n \equiv 0(\text{mod } 4) \\ \frac{2n}{3} & ;\text{if } n \equiv 0(\text{mod } 3)\ \&\ n \not\equiv 0(\text{mod } 4) \\ n & ;\text{otherwise.} \end{cases}$$

Theorem 2.7 $\gamma_{cd}(P_n \times K_1) = n$.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n\}$ be the set of vertices of $C_n \times K_1$ with $|V(G)| = 2n$. Here v_1 *and* v_n are the vertices of degree 2, $v_2, v_3, \dots, v_{n-1}$ are the vertices of degree 3 and $u_1, u_2, \dots, u_n$ are the vertices of degree 1.
Therefore, $\sum_{v\in V(G)} d(v) = 2(2) + 3(n-2) + n = 2(2n-1)$.
If n is even, then define $D = \{v_{2k} \cup u_{2k+1}/0 \leqslant k \leqslant \frac{n}{2}\}$ with $|D| = n$.
If n is odd, then define $D = \left\{v_{2k} \cup u_{2k+1}/0 \leqslant k \leqslant \frac{n-1}{2}\right\}$ with $|D| = n$.
Then, in both the cases D is a minimal dominating set with minimum cardinality as $\gamma(P_n \times K_1) = n$.
Now, if n is even then there are $\left(\frac{n}{2} - 1\right)$ vertices of degree 3, one vertex of degree 2 and $\frac{n}{2}$ vertices of degree 1 and so, $\sum_{v\in D} d(v) = 3\left(\frac{n}{2} - 1\right) + 2 + \frac{n}{2} = 2n - 1$.

Then, D satisfies Condition (1) to be a congruent dominating set.
Also, if n is odd then there is $\left(\frac{n-1}{2}\right)$ vertices of degree 3 and $\left(\frac{n+1}{2}\right)$ vertices of degree 1 and so, $\sum_{v\in D} d(v) = 3\left(\frac{n-1}{2}\right) + \frac{n+1}{2} = 2n-1$.
Then, in both cases, D satisfies Condition (1) to be a congruent dominating set. Therefore, D is a congruent dominating set.
Since, D is a minimal dominating set with minimum cardinality, it is a minimal congruent dominating set with minimum cardinality.
Hence, $\gamma_{cd}(P_n \times K_1) = n$.

Theorem 2.8 For $n \geq 3$,

$$\gamma_{cd}(C_n \times P_2) = \begin{cases} \frac{n}{2} & ;\text{if } n \equiv 0(\text{mod}4) \\ \frac{2n}{3} & ;\text{if } n \equiv 0(\text{mod}3) \text{ and } n \not\equiv 0(\text{mod}4) \\ n & ;\text{otherwise} \end{cases}$$

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$ be the set of vertices of $C_n \times P_2$ with $|V(G)| = 2n$.
Here, $d(v) = 3\ ;\forall v \in V(G)$. Therefore, $\sum_{v\in V(G)} d(v) = 6n$.

Case-1: $n \equiv 0(\text{mod}4)$
Define $D = \left\{v_{4k+3} \cup u_{4k+1} / 0 \leqslant k \leqslant \frac{n}{4} - 1\right\}$ with $|D| = \frac{n}{2}$. Then, D is a minimal dominating set with minimum cardinality as for $n \equiv 0(\text{mod}4)$, $\gamma(C_n \times P_2) = \frac{n}{2}$.
Moreover, $\sum_{v\in D} d(v) = \frac{3n}{2}$. Then, D satisfies Condition (1) to be a congruent dominating set. Therefore, D is congruent dominating set.
Since, D is a minimal dominating set with minimum cardinality, it is a minimal congruent dominating set with minimum cardinality.

Case-2: $n \equiv 0(\text{mod } 3)$ and $n \not\equiv 0(\text{mod } 4)$
Define $D = \left\{v_{3k+2} \cup u_{3k+2} / 0 \leqslant k \leqslant \frac{n}{3} - 1\right\}$ with $|D| = \frac{2n}{3}$. Then D is minimal dominating set with as for any $u \in D$, $D - \{u\}$ doesn't dominate $N(u)$.
Moreover, $\sum_{v\in D} d(v) = 3\left(\frac{2n}{3}\right) = 2n$. Then, D satisfies Condition (1) to be a congruent dominating set.
Therefore, D is a minimal congruent dominating set, as it is a minimal dominating set.
Now, for $n \not\equiv 0(\text{mod}4)$, $\gamma(C_n \times P_2) > \frac{n}{2}$ and so $\sum_{v\in D} d(v) > \frac{3n}{2}$.
Let $D' \subset V(G)$ be the set of vertices of $C_n \times P_2$ with $|D'| < |D|$ and $\sum_{v\in D'} d(v) = t$.
Then there does not exist any $t \in \mathbb{N}$ such that $\frac{3n}{2} < t < 2n$ and $t|6n$.
Therefore, D is minimal congruent dominating set with minimal cardinality.

Case-3: $n \not\equiv 0(\text{mod } 3{,}4)$
Define $D = \{v_1, v_2, \cdots v_n\}$ with $|D| = n$.
Then, D is a minimal dominating set with as for any $u \in D$, $D - \{u\}$ doesn't dominate u.
Moreover, $\sum_{v\in D} d(v) = 3n$. Then, D satisfies Condition (1) to be a congruent dominating set.
Therefore, D is minimal congruent dominating set, as it is minimal dominating set.
Now, for $n \not\equiv 0(\text{mod}4)$, $\gamma(C_n \times P_2) > \frac{n}{2}$ and so $\sum_{v\in D} d(v) > \frac{3n}{2}$.
Let $D' \subset V(G)$ be the set of vertices of $C_n \times P_2$ with $|D'| < |D|$ and $\sum_{v\in D'} d(v) = t$.
Then, there does not exist any $t \in \mathbb{N}$ such that $\frac{3n}{2} < t < 2n$ and $t|6n$.
Therefore, D is a minimal congruent dominating set with minimal cardinality.
Hence,

$$\gamma_{cd}(C_n \times P_2) = \begin{cases} \frac{n}{2} & ;\text{if } n \equiv 0(\text{mod}4) \\ \frac{2n}{3} & ;\text{if } n \equiv 0(\text{mod}3) \text{ and } n \not\equiv 0(\text{mod}4) \\ n & ;\text{Otherwise.} \end{cases}$$

III. CONCLUSION

The concept of congruent domination in graphs has been recently introduced by Vaidya and Vadhel [13] and further investigated in [14, 15, 16]. The concept is a frontier between number theory and theory of graphs. The congruent domination numbers have been investigated for the graphs obtained by means of some graph operations.

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