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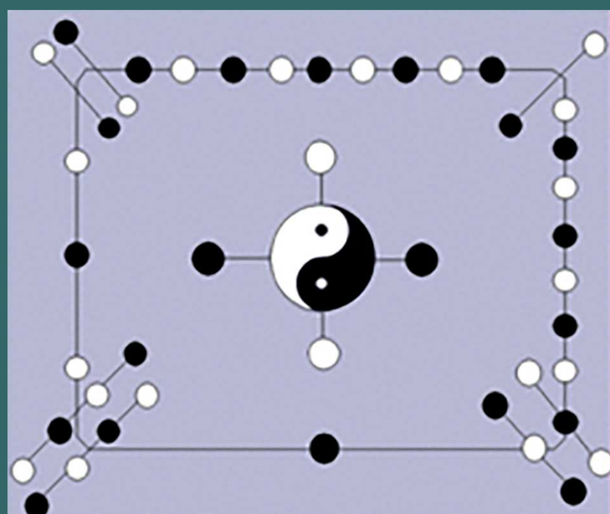

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A man can fail many times, but he isn't a failure until he begins to blame somebody else and stops trying.

By J.Burroughs , an American essayist and naturalist.

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Computing Zagreb Polynomials of Generalized xyz-Point-Line Transformation Graphs $T^{xyz}(G)$ With $z = -$

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Abstract: In this paper, we discuss relations among Zagreb polynomials of a graph G and generalized xyz-point-line transformation graphs $T^{xyz}(G)$ when $z = -$. Zagreb polynomials of xyz-point-line transformation graphs are obtained in terms of Zagreb polynomials of the graph G.

Key Words: Zagreb indices, Zagreb polynomials, xyz-point-line transformation graphs.

AMS(2010): 05C07, 05C31.

§1. Introduction

By a Graph $G = (V, E)$ we mean a nontrivial, finite, simple, undirected graph with vertex set V and an edge set E of order n and size m. The *degree* $d_G(v)$ of a vertex v in G is the number of edges incident to it in G. Let $\overline{G}$, $L(G)$ and $S(G)$ of a graph G are complement, line graph and subdivision graph of a graph G respectively. The *partial complement of subdivision graph* $\overline{S}(G)$ of a graph G whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and the other is an edge of G non incident with it.

In this paper, we denote $u \sim v$ ($u \nsim v$) for vertices u and v are adjacent (resp., nonadjacent), $e \sim f$ ($e \nsim f$) for the adjacent (resp., nonadjacent) edges e and f and $u \sim e$ ($u \nsim e$) for the vertex u and an edge e are incident (resp., nonincident) in G. Other undefined notations and terminologies can be found in [17] or [19].

Polynomials are one of the graph invariants which does not depend on the labeling or pictorial representation of the graph. A topological index is also one such graph invariant. The topological indices have their applications in several branches of science and technology.

The first and second Zagreb indices are amongst the oldest and best known topological indices defined in 1972 by Gutman [15] as follows:

$$M_1(G) = \sum_{v \in V(G)} d_G(v)^2 \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v),$$

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respectively. These are widely studied degree based topological indices due to their applications in chemistry, for details refer to [10,11,14,16,23]. The first Zagreb index [21] can also be expressed as

$$M_1(G) = \sum_{uv\in E(G)} [d_G(u) + d_G(v)].$$

Ashrafi et al. [1] defined respectively the first and second Zagreb coindices as

$$\overline{M_1}(G) = \sum_{uv\notin E(G)} [d_G(u) + d_G(v)] \text{ and } \overline{M_2}(G) = \sum_{uv\notin E(G)} [d_G(u)d_G(v)].$$

In 2004, Milićević et al. [20] reformulated the Zagreb indices in terms of edge-degrees instead of vertex-degrees. The first and second reformulated Zagreb indices are defined respectively by

$$EM_1(G) = \sum_{e\in E(G)} d_G(e)^2 \text{ and } EM_2(G) = \sum_{e\sim f}[d_G(e)d_G(f)].$$

In [18], Hosamani and Trinajstić defined the first and second reformulated Zagreb coindices respectively as

$$\overline{EM_1}(G) = \sum_{e\nsim f}[d_G(e) + d_G(f)] \text{ and } \overline{EM_2}(G) = \sum_{e\nsim f}[d_G(e)d_G(f)].$$

Considering the Zagreb indices, Fath-Tabar [13] defined first and the second *Zagreb polynomials* as

$$M_1(G,x) = \sum_{v_i,v_j\in E(G)} x^{d_G(v_i)+d_G(v_j)} \text{ and } M_2(G,x) = \sum_{v_i,v_j\in E(G)} x^{d_G(v_i)\cdot d_G(v_j)}$$

respectively, where x is a variable. In addition, Shuxian [22] defined two polynomials related to the first Zagreb index in the form

$$M_1^*(G,x) = \sum_{v_i\in V(G)} d_G(v_i)x^{d_G(v_i)} \text{ and } M_0(G,x) = \sum_{v_i\in V(G)} x^{d_G(v_i)}.$$

A. R. Bindusree et al. defined the following polynomials in [9],

$$M_4(G,x) = \sum_{v_i,v_j\in E(G)} x^{d_G(v_i)((d_G(v_i)+d_G(v_j))},\ M_5(G,x) = \sum_{v_i,v_j\in E(G)} x^{d_G(v_j)((d_G(v_i)+d_G(v_j))},$$
$$M_{a,b}(G,x) = \sum_{v_i,v_j\in E(G)} x^{ad_G(v_i)+bd_G(v_j)},\ M'_{a,b}(G,x) = \sum_{v_i,v_j\in E(G)} x^{(d_G(v_i)+a)(d_G(v_j)+b)}.$$

§2. Generalized xyz-Point-Line Transformation Graph $T^{xyz}(G)$

For a graph $G = (V,E)$, let G^0 be the graph with $V(G^0) = V(G)$ and with no edges, G^1 the complete graph with $V(G^1) = V(G)$, $G^+ = G$, and $G^- = \overline{G}$. Let $\mathcal{G}$ denotes the set of simple graphs. The graph operations depending on $x,y,z \in \{0,1,+,-\}$ induce functions $T^{xyz} : \mathcal{G} \to \mathcal{G}$. These operations were introduced by Deng et al. in [12] and named them as xyz-transformations of G, denoted by $T^{xyz}(G) = G^{xyz}$. In [2], Wu Bayoindureng et al. introduced the total transformation graphs and studied their basic properties. Motivated by this, Basavanagoud [3] studied the basic properties of the xyz-transformation graphs by changing them as *xyz-point-line transformation graphs* and denoted as $T^{xyz}(G)$ to avoid confusion between various transformations.

Definition 2.1([12]) *Given a graph G with vertex set $V(G)$ and edge set $E(G)$ and three variables $x, y, z \in \{0, 1, +, -\}$, the xyz-point-line transformation graph $T^{xyz}(G)$ of G is the graph with vertex set $V(T^{xyz}(G)) = V(G) \cup E(G)$ and the edge set $E(T^{xyz}(G)) = E((G)^x) \cup E((L(G))^y) \cup E(W)$ where $W = S(G)$ if $z = +$, $W = \overline{S}(G)$ if $z = -$, W is the graph with $V(W) = V(G) \cup E(G)$ and with no edges if $z = 0$ and W is the complete bipartite graph with parts $V(G)$ and $E(G)$ if $z = 1$.*

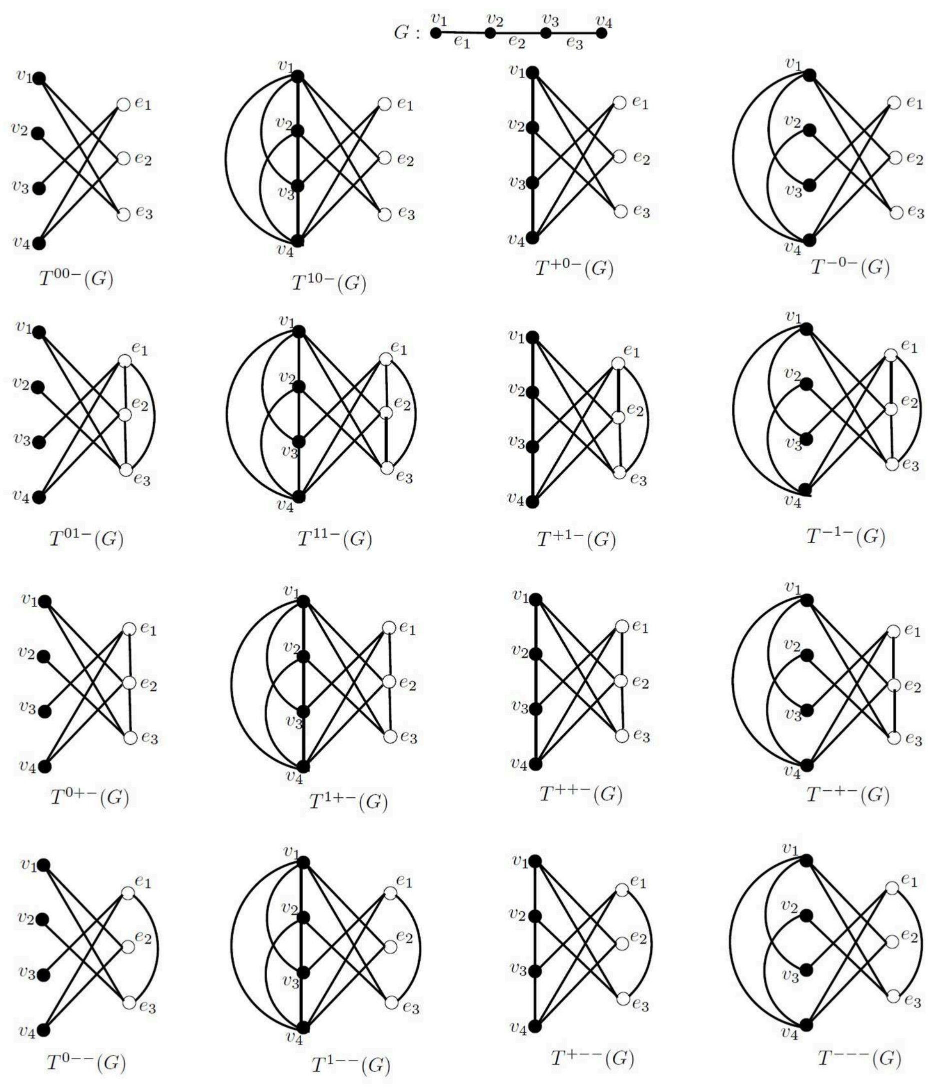

Figure 1. P_4 and its generalized xyz-polint-line transformation graphs $T^{xy-}(P_4)$.

Since there are 64 distinct 3 - permutations of $\{0, 1, +, -\}$. Thus 64 kinds of generalized xyz-point-line transformation graphs are obtained. There are 16 different graphs for each case when $z = 0$, $z = 1$, $z = +$, $z = -$. In this paper, we consider the xyz-point-line transformation graph $T^{xyz}(G)$ with $z = -$. The self-explanatory examples of the path P_4 and its *xyz-point-line transformation graphs* $T^{xy-}(P_4)$ are depicted in Figure 1. For more on generalized transformation graphs refer to [2]-[8].

The following Observations are useful in proving the theorems.

Observation 2.1([4]) Let G be a graph of order n and size m. Let v be a vertex of G and $Y = \{0, 1, +, -\}$. Then

$$d_{T^{xy-}}(v) = \begin{cases} m - d_G(v) & if\ x = 0, y \in Y, \\ n + m - 1 - d_G(v) & if\ x = 1, y \in Y, \\ m & if\ x = +, y \in Y, \\ n + m - 1 - 2d_G(v) & if\ x = -, y \in Y. \end{cases}$$

Observation 2.2([4]) Let G be a graph of order n and size m. Let e be an edge of G and $Y = \{0, 1, +, -\}$. Then

$$d_{T^{xy-}}(e) = \begin{cases} n - 2 & if\ y = 0, x \in Y, \\ n + m - 3 & if\ y = 1, x \in Y, \\ n - 2 + d_G(e) & if\ y = +, x \in Y, \\ n + m - 3 - d_G(e) & if\ y = -, x \in Y. \end{cases}$$

§3. Results on the Zagreb Polynomials of $T^{xy-}(G)$

In this section, we obtain the Zagreb polynomials of the xyz-point-line transformation graph $T^{xyz}(G)$ with $z = -$. In this process, to cover the edges in the complements $\overline{G}, \overline{S}(G)$ and $\overline{L(G)}$ we need the degrees of nonadjacent vertices (or edges) in the graph. Degrees of these nonadjacent vertices (or edges) gives Zagreb coindices. To overcome from this problem Basavanagoud and Jakkannavar [7] defined the first, second and third Zagreb co-polynomials of a graph G by using the concept of Zagreb coindices as

$$\overline{M_1}(G, x) = \sum_{v_i, v_j \notin E(G)} x^{d_G(v_i) + d_G(v_j)}, \ \ \overline{M_2}(G, x) = \sum_{v_i, v_j \notin E(G)} x^{d_G(v_i) \cdot d_G(v_j)}$$

and

$$\overline{M_3}(G, x) = \sum_{v_i, v_j \notin E(G)} x^{|d_G(v_i) - d_G(v_j)|}$$

respectively, where x is a variable.

In addition, in [7] they defined

$$\overline{M_4}(G,x) = \sum_{v_i,v_j \notin E(G)} x^{d_G(v_i)(d_G(v_i)+d_G(v_j))}, \quad \overline{M_5}(G,x) = \sum_{v_i,v_j \notin E(G)} x^{d_G(v_j)(d_G(v_i)+d_G(v_j))},$$
$$\overline{M_{a,b}}(G,x) = \sum_{v_i,v_j \notin E(G)} x^{ad_G(v_i)+bd_G(v_j)}, \quad \overline{M'}_{a,b}(G,x) = \sum_{v_i,v_j \notin E(G)} x^{(d_G(v_i)+a)(d_G(v_j)+b)}.$$

The following theorems give results on Zagreb polynomials of the generalized xyz-point-line transformation graphs $T^{xy-}(G)$.

Theorem 3.1 *Let G be a graph of order n and size m. Then Zagreb polynomials of $T^{00-}(G)$ are*

$$\begin{aligned}
M_1(T^{00-}(G),x) &= mx^{m+n-2}M_0(G,x^{-1}) - x^{m+n-2}M_1^*(G,x^{-1})\\
M_2(T^{00-}(G),x) &= mx^{m(n-2)}M_0(G,x^{-(n-2)}) - x^{m(n-2)}M_1^*(G,x^{-(n-2)})\\
M_3(T^{00-}(G),x) &= mx^{|n-m-2|}M_0(G,x) - x^{|n-m-2|}M_1^*(G,x).
\end{aligned}$$

Proof From Observations (2.1) and (2.2) we have

$$d_{T^{00-}(G)}(v) = \begin{cases} m - d_G(v) & if\ v \in V(G), \\ n-2 & if\ v \in E(G). \end{cases}$$

By using definition of $M_1(G,x)$, we have

$$\begin{aligned}
M_1(T^{00-}(G),x) &= \sum_{uv \in E(T^{00-}(G))} x^{d_{T^{00-}(G)}(u)+d_{T^{00-}(G)}(v)}\\
&= \sum_{u \nsim v} x^{d_{T^{00-}(G)}(u)+d_{T^{00-}(G)}(v)} = \sum_{u \nsim v} x^{m-d_G(v)+n-2}\\
&= x^{2-m-n} \sum_{v \in V(G)} (m-d_G(v))x^{-d_G(u)}.\\
&= mx^{m+n-2}M_0(G,x^{-1}) - x^{m+n-2}M_1^*(G,x^{-1}).
\end{aligned}$$

By using definition of $M_2(G,x)$, we have

$$\begin{aligned}
M_2(T^{00-}(G),x) &= \sum_{uv \in E(T^{00-}(G))} x^{d_{T^{00-}(G)}(u)d_{T^{00-}(G)}(v)}\\
&= \sum_{u \nsim v} x^{d_{T^{00-}(G)}(u)d_{T^{00-}(G)}(v)}\\
&= \sum_{u \nsim v} x^{(m-d_G(u))(n-2)}\\
&= x^{-m(n-2)} \sum_{v \in V(G)} (m-d_G)x^{-m(n-2)}d_G(v)\\
&= mx^{m(n-2)}M_0(G,x^{-(n-2)}) - x^{m(n-2)}M_1^*(G,x^{-(n-2)})
\end{aligned}$$

By using definition of $M_3(G,x)$, we have

$$\begin{aligned}
M_3(T^{00-}(G),x) &= \sum_{uv\in E(T^{00-}(G))} x^{|d_{T^{00-}(G)}(u)-d_{T^{00-}(G)}(v)|}\\
&= \sum_{u\nsim v} x^{|d_{T^{00-}(G)}(u)-d_{T^{00-}(G)}(v)|}\\
&= \sum_{u\nsim v} x^{|m-d_G(u)-n+2|}\\
&= x^{|n-m-2|}\sum_{u\in V(G)} (m-d_G(v))x^{|d_G(u)|}\\
&= mx^{|n-m-2|}M_0(G,x)-x^{|n-m-2|}M_1^*(G,x) \qquad \square
\end{aligned}$$

Theorem 3.2 *Let G be a graph of order n and size m. Then, the Zagreb polynomials of $T^{01-}(G)$ are*

$$\begin{aligned}
M_1(T^{01-}(G),x) &= \binom{m}{2}x^{2m+n-3}+mx^{2m+n-3}M_0(G,x^{-1})-x^{2m+n-3}M_1^*(G,x^{-1})\\
M_2(T^{01-}(G),x) &= \binom{m}{2}x^{(n+m-3)^2}+mx^{m(n+m-3)}M_0(G,x^{-(n+m-3)})\\
&\quad -x^{m(n+m-3)}M_1^*(G,x^{-(n+m-3)})\\
M_3(T^{01-}(G),x) &= \binom{m}{2}+mx^{n-3}M_0(G,x^{-1})-x^{n-3}M_1^*(G,x^{-1}).
\end{aligned}$$

Proof From Observations (2.1) and (2.2) we have

$$d_{T^{01-}(G)}(v)=\begin{cases} m-d_G(v) & if\ v\in V(G)\\ n+m-3 & if\ v\in E(G)\end{cases}$$

By using the definition of $M_1(G,x)$, we have

$$\begin{aligned}
M_1(T^{01-}(G),x) &= \sum_{uv\in E(T^{01-}(G))} x^{d_{T^{01-}(G)}(u)+d_{T^{01-}(G)}(v)}\\
&= \sum_{uv\in E(L(G))} x^{d_{T^{01-}(G)}(u)+d_{T^{01-}(G)}(v)}\\
&\quad + \sum_{u,v\notin E(LG)} x^{d_{T^{01-}(G)}(u)+d_{T^{01-}(G)}(v)}\\
&\quad + \sum_{u\nsim v} x^{d_{T^{01-}(G)}(u)+d_{T^{01-}(G)}(v)}\\
&= \sum_{uv\in E(L(G))} x^{2(n+m-3)}+\sum_{u,v\notin E(L(G)} x^{2(n+m-3)}+\sum_{u\nsim v} x^{m-d_G(v)+n+m-3}\\
&= \binom{m}{2}x^{2m+n-3}+mx^{2m+n-3}M_0(G,x^{-1})-x^{2m+n-3}M_1^*(G,x^{-1}).
\end{aligned}$$

Similarly, we known

$$\begin{aligned}
M_2(T^{01-}(G),x) &= \sum_{uv\in E(T^{01-}(G))} x^{d_{T^{01-}(G)}(u)d_{T^{01-}(G)}(v)}\\
&= \sum_{uv\in E(L(G))} x^{d_{T^{01-}(G)}(u)d_{T^{01-}(G)}(v)} + \sum_{u,v\notin E(LG)} x^{d_{T^{01-}(G)}(u)d_{T^{01-}(G)}(v)}\\
&\quad + \sum_{u\nsim v} x^{d_{T^{01-}(G)}(u)d_{T^{01-}(G)}(v)}\\
&= \binom{m}{2} x^{(n+m-3)^2} + mx^{m(n+m-3)}M_0(G,x^{-(n+m-3)})\\
&\quad - x^{m(n+m-3)}M_1^*(G,x^{-(n+m-3)})
\end{aligned}$$

$$\begin{aligned}
M_3(T^{01-}(G),x) &= \sum_{uv\in E(T^{01-}(G))} x^{|d_{T^{01-}(G)}(u)-d_{T^{01-}(G)}(v)|}\\
&= \sum_{uv\in E(L(G))} x^{|d_{T^{01-}(G)}(u)-d_{T^{01-}(G)}(v)|} + \sum_{u,v\notin E(L(G)} x^{|d_{T^{01-}(G)}(u)-d_{T^{01-}(G)}(v)|}\\
&\quad + \sum_{u\nsim v} x^{|d_{T^{01-}(G)}(u)-d_{T^{01-}(G)}(v)|}\\
&= \binom{m}{2} + mx^{n-3}M_0(G,x^{-1}) - x^{n-3}M_1^*(G,x^{-1}).
\end{aligned}$$

□

Theorem 3.3 *Let G be a graph of order n and size m. Then Zagreb polynomials of $T^{0+-}(G)$ are*

$$\begin{aligned}
M_1(T^{0+-}(G),x) &= x^{2m}M_1(L(G),x^{-1}) + x^{m+n-2}\sum_{u\nsim v} x^{-d_G(u)+d_G(v)}\\
M_2(T^{0+-}(G),x) &= M'_{(n-2,n-2)}(L(G),x) + \sum_{u\nsim v} x^{(m-d_G(u))(n-2+d_G(v))}\\
M_3(T^{0+-}(G),x) &= M_3(L(G),x) - x^{m-n+2}\sum_{u\nsim v} x^{|d_G(u)+d_G(v)}
\end{aligned}$$

Proof From Observations (2.1) and (2.2) we have

$$d_{T^{0+-}(G)}(v) = \begin{cases} m-d_G(v) & if\ v\in V(G)\\ n-2+d_G(v) & if\ v\in E(G)\end{cases}$$

Applying the definition of $M_1(G,x)$, we have

$$\begin{aligned}
M_1(T^{0+-}(G),x) &= \sum_{uv\in E(T^{0+-}(G))} x^{d_{T^{0+-}(G)}(u)+d_{T^{0+-}(G)}(v)}\\
&= \sum_{uv\in E(L(G))} x^{d_{T^{0+-}(G)}(u)+d_{T^{0+-}(G)}(v)} + \sum_{u\nsim v} x^{d_{T^{0+-}(G)}(u)+d_{T^{0+-}(G)}(v)}\\
&= x^{2m}M_1(L(G),x^{-1}) + x^{m+n-2}\sum_{u\nsim v} x^{-d_G(u)+d_G(v)}.
\end{aligned}$$

Similarly, we know

$$\begin{aligned} M_2(T^{0+-}(G),x) &= \sum_{uv\in E(T^{0+-}(G))} x^{d_{T^{0+-}(G)}(u)d_{T^{0+-}(G)}(v)} \\ &= \sum_{uv\in E(L(G))} x^{d_{T^{0+-}(G)}(u)d_{T^{0+-}(G)}(v)} + \sum_{u\nsim v} x^{d_{T^{0+-}(G)}(u)d_{T^{0+-}(G)}(v)} \\ &= M'_{(n-2,n-2)}(L(G),x) + \sum_{u\nsim v} x^{(m-d_G(u))(n-2+d_G(v))} \end{aligned}$$

and

$$\begin{aligned} M_3(T^{0+-}(G),x) &= \sum_{uv\in E(T^{0+-}(G))} x^{|d_{T^{0+-}(G)}(u)-d_{T^{0+-}(G)}(v)|} \\ &= \sum_{uv\in E(L(G))} x^{|d_{T^{0+-}(G)}(u)-d_{T^{0+-}(G)}(v)|} + \sum_{u\nsim v} x^{|d_{T^{0+-}(G)}(u)-d_{T^{0+-}(G)}(v)|} \\ &= M_3(L(G),x) - x^{m-n+2}\sum_{u\nsim v} x^{|d_G(u)+d_G(v)|}. \qquad \square \end{aligned}$$

Theorem 3.4 *Let G be a graph of order n and size m. Then Zagreb polynomials of $T^{0--}(G)$ are*

$$\begin{aligned} M_1(T^{0--}(G),x) &= x^{2(n+m-3)}\overline{M_1}(L(G),x^{-1}) + x^{2m+n-3}\sum_{u,v\notin E(G)} x^{-(d_G(u)+d_G(v))} \\ M_2(T^{0--}(G),x) &= M'_{(-n-m+3,-n-m+3)}(L(G),x) + \sum_{u\nsim v} x^{(m-d_G(u))(m+n-3-d_G(v))} \\ M_3(T^{0--}(G),x) &= M_3(L(G),x) - x^{|n-3|}\sum_{u\nsim v} x^{|d_G(u)+d_G(v)} \end{aligned}$$

Proof From Observations (2.1) and (2.2) we have

$$d_{T^{0--}(G)}(v) = \begin{cases} m-d_G(v) & if\ v\in V(G) \\ n+m-3-d_G(v) & if\ v\in E(G) \end{cases}$$

By the definition of $M_1(G,x)$, we have

$$\begin{aligned} M_1(T^{0--}(G),x) &= \sum_{uv\in E(T^{0--}(G))} x^{d_{T^{0--}(G)}(u)+d_{T^{0+-}(G)}(v)} \\ &= \sum_{uv\notin E(L(G))} x^{d_{T^{0--}(G)}(u)+d_{T^{0--}(G)}(v)} + \sum_{u\nsim v} x^{d_{T^{0--}(G)}(u)+d_{T^{0--}(G)}(v)} \\ &= x^{2(n+m-3)}\overline{M_1}(L(G),x^{-1}) + x^{2m+n-3}\sum_{u,v\notin E(G)} x^{-(d_G(u)+d_G(v))} \end{aligned}$$

Similarly, we know

$$M_2(T^{0--}(G),x) = \sum_{uv\in E(T^{0--}(G))} x^{d_{T^{0--}(G)}(u)d_{T^{0--}(G)}(v)}$$

$$
\begin{aligned}
&= \sum_{uv\notin E(L(G))} x^{d_{T^{0--}(G)}(u)d_{T^{0--}(G)}(v)} + \sum_{u\nsim v} x^{d_{T^{0--}(G)}(u)d_{T^{0--}(G)}(v)} \\
&= M'_{(-n-m+3,-n-m+3)}(L(G),x) + \sum_{u\nsim v} x^{(m-d_G(u))(m+n-3-d_G(v))}.
\end{aligned}
$$

$$
\begin{aligned}
M_3(T^{0--}(G),x) &= \sum_{uv\in E(T^{0--}(G))} x^{|d_{T^{0--}(G)}(u)-d_{T^{0--}(G)}(v)|} \\
&= \sum_{uv\notin E(L(G))} x^{|d_{T^{0--}(G)}(u)-d_{T^{0--}(G)}(v)|} + \sum_{u\nsim v} x^{|d_{T^{0--}(G)}(u)-d_{T^{0--}(G)}(v)|} \\
&= M_3(L(G),x) - x^{|n-3|}\sum_{u\nsim v} x^{|d_G(u)+d_G(v)|}. \qquad \square
\end{aligned}
$$

Theorem 3.5 *Let G be a graph of order n and size m. Then*

$$
\begin{aligned}
M_1(T^{10-}(G),x) &= x^{2(n+m-1)}M_{-1,-1}(G,x) + x^{2m}M_1(\overline{G},x) + mx^{(2n+m-3)}M_0(G,x^{-1}) \\
&\quad + M_1^*(G,x^{-1})x^{(2n+m-3)} \\
M_2(T^{10-}(G),x) &= M'_{-(n+m-1),-(n+m-1)}(G,x) + M'_{m,m}(\overline{G},x) \\
&\quad + mx^{(n+m-1)(n+2)}M_0(G,x^{-(n+2)}) - x^{(n+m-1)(n+2)}M_1^*(G,x^{-(n+2)}) \\
M_3(T^{10-}(G),x) &= M_3(G,x) + \overline{M_3}(G,x) + mx^{m+1}M_0(G,x^{-1}) - x^{m+1}M_1^*(G,x^{-1}).
\end{aligned}
$$

Proof From Observations (2.1) and (2.2) we have,

$$
d_{T^{10-}(G)}(v) = \begin{cases} n+m-1+d_G(v) & if\ v\in V(G) \\ n-2 & if\ v\in E(G) \end{cases}
$$

By the definition of $M_1(G,x)$, we know

$$
\begin{aligned}
M_1(T^{10-}(G),x) &= \sum_{uv\in E(T^{10-}(G))} x^{d_{T^{10-}(G)}(u)+d_{T^{10-}(G)}(v)} \\
&= \sum_{uv\in E(G)} x^{d_{T^{10-}(G)}(u)+d_{T^{10-}(G)}(v)} + \sum_{uv\notin E(G)} x^{d_{T^{10-}(G)}(u)+d_{T^{10-}(G)}(v)} \\
&\quad + \sum_{u\nsim v} x^{d_{T^{10-}(G)}(u)+d_{T^{10-}(G)}(v)} \\
&= \sum_{uv\in E(G)} x^{n+m-1+d_G(u)+n+m-1+d_G(v)} + \sum_{uv\notin E(G)} x^{n+m-1+d_G(u)+n+m-1+d_G(v)} \\
&\quad + \sum_{u\nsim v} x^{n+m-1-d_G(v)+n-2} \\
&= x^{2(n+m-1)}M_{-1,-1}(G,x) + x^{2m}M_1(\overline{G},x) + mx^{(2n+m-3)}M_0(G,x^{-1}) \\
&\quad + M_1^*(G,x^{-1})x^{(2n+m-3)}.
\end{aligned}
$$

Similarly, by the definition of $M_2(G,x)$, we have

$$
\begin{aligned}
M_2(T^{10-}(G),x) &= \sum_{uv\in E(T^{10-}(G))} x^{d_{T^{10-}(G)}(u)d_{T^{10-}(G)}(v)} \\
&= \sum_{uv\in E(G)} x^{d_{T^{10-}(G)}(u)d_{T^{10-}(G)}(v)} + \sum_{uv\notin E(G)} x^{d_{T^{10-}(G)}(u)d_{T^{10-}(G)}(v)} \\
&\quad + \sum_{u\nsim v} x^{d_{T^{10-}(G)}(u)d_{T^{10-}(G)}(v)} \\
&= M'_{-(n+m-1),-(n+m-1)}(G,x) + M'_{m,m}(\overline{G},x) \\
&\quad + mx^{(n+m-1)(n+2)}M_0(G,x^{-(n+2)}) - x^{(n+m-1)(n+2)}M_1^*(G,x^{-(n+2)}).
\end{aligned}
$$

$$
\begin{aligned}
M_3(T^{10-}(G),x) &= \sum_{uv\in E(T^{10-}(G))} x^{|d_{T^{10-}(G)}(u)-d_{T^{10-}(G)}(v)|} \\
&= \sum_{uv\in E(G)} x^{|d_{T^{10-}(G)}(u)-d_{T^{10-}(G)}(v)|} + \sum_{uv\notin E(G)} x^{|d_{T^{10-}(G)}(u)-d_{T^{10-}(G)}(v)|} \\
&\quad + \sum_{u\nsim v} x^{|d_{T^{10+}(G)}(u)-d_{T^{10+}(G)}(v)|} \\
&= M_3(G,x) + \overline{M_3}(G,x) + mx^{m+1}M_0(G,x^{-1}) - x^{m+1}M_1^*(G,x^{-1}). \qquad \square
\end{aligned}
$$

The proof of following theorems are analogous to that of Theorems 3.1-3.5.

Theorem 3.6 *Let G be a graph of order n and size m. Then*

$$
\begin{aligned}
M_1(T^{11-}(G),x) &= x^{2(n+m-1)}M_1(G,x^{-1}) + x^{2m}M_1(\overline{G},x) \\
&\quad + \binom{m}{2}x^{2(n+m-3)} + x^{2(n+m-3)}M_1^*(G,x^{-1}) \\
M_2(T^{11-}(G),x) &= M'_{-(n+m-1),-(n+m-1)}(G,x) + M'_{m,m}(\overline{G},x) + \binom{m}{2}x^{(n+m-3)^2} \\
&\quad + x^{(n+m-1)(n+m-3)}M_1^*(G,x^{(n+m-3)}) \\
M_3(T^{11-}(G),x) &= M_1(G,x) + \overline{M_1}(G,x) + \binom{m}{2} + x^2M_1^*(G,x).
\end{aligned}
$$

Theorem 3.7 *Let G be a graph of order n and size m. Then*

$$
\begin{aligned}
M_1(T^{1+-}(G),x) &= x^{2(n+m-1)}M_{-1,-1}(G,x) + x^{2m}M_1(\overline{G},x) \\
&\quad + x^{2(n-2)}M_1(L(G),x) + x^{2(n+m-3)}\sum_{u\nsim v} x^{-d_G(u)+d_G(v)} \\
M_2(T^{1+-}(G),x) &= M'_{-(n+m-1),(n-2)}(G,x^{-1}) + \overline{M'}_{-(n+m-1),(n-2)}(G,x^{-1}) \\
&\quad + M'_{n-2,n-2}(L(G),x) + \sum_{u\nsim v} x^{(n+m-1-d_G(u))(n-2+d_G(v))} \\
M_3(T^{1+-}(G),x) &= x^{m+1}M_3(G,x) + x^{2m}\overline{M_1}(G,x) + M_3(L(G),x) + x^{m+1}\sum_{u\nsim v} x^{|d_G(u)+d_G(v)|}.
\end{aligned}
$$

Theorem 3.8 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{1--}(G),x) &= x^{2(n+m-1)}M_{-1,-1}(G,x) + x^{2m}M_1(\overline{G},x) + x^{2(n+m-3)}\overline{M}_1(L(G),x) \\
&\quad + x^{2(n+m-2)}\sum_{u \nsim v} x^{-d_G(u)-d_G(v)} \\
M_2(T^{1--}(G),x) &= M'_{-(n+m-1),-(n+m-1)}(G,x) + M'_{m,m}(\overline{G},x) + M'_{n+2,n+2}(L(\overline{G}),x) \\
&\quad + \sum_{u \nsim e} x^{(n+m-1-d_G(u))(n+m-3-d_G(v))} \\
M_3(T^{1--}(G),x) &= M_1(G,x) + \overline{M_1}(G,x) + \overline{M_1}(L(G),x) + x^2\sum_{u \nsim v} x^{|d_G(u)+d_G(v)|}.
\end{aligned}$$

Theorem 3.9 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{+0-}(G),x) &= mx^{2m} + m(n-2)x^{m+n-2} \\
M_2(T^{+0-}(G),x) &= mx^{m^2} + m(n-2)x^{m(n-2)} \\
M_3(T^{+0-}(G),x) &= m + m(n-2)x^{|m-n+2|}.
\end{aligned}$$

Theorem 3.10 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{+1-}(G),x) &= mx^{2m} + \binom{m}{2}x^{2(n+m-3)} + m(n-2)x^{(2m+n-3)} \\
M_2(T^{+1-}(G),x) &= mx^{m^2} + \binom{m}{2}x^{(n+m-3)^2} + m(n-2)x^{m(m+n-3)} \\
M_3(T^{+1-}(G),x) &= m + \binom{m}{2} + m(n-2)x^{|n-3|}.
\end{aligned}$$

Theorem 3.11 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{++-}(G),x) &= mx^{2m} + x^{2(n-2)}\overline{M}_1(L(G),x) + mx^{m+n-2}M_0(G,x) - x^{m+n-2}M_1^*(G,x) \\
M_2(T^{++-}(G),x) &= mx^{m^2} + M_{n-2,n-2}(L(G),x) + mx^{m(n-2)}M_0(G,x^m) - x^{m(n-2)}M_1^*(G,x^m) \\
M_3(T^{++-}(G),x) &= m + M_3(L(G),x) + mx^{|m+n-2|}M_0(G,x) - x^{|m+n-2|}M_1^*(G,x).
\end{aligned}$$

Theorem 3.12 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{+--}(G),x) &= mx^{2m} + x^{2(n+m-3)}\overline{M}_1(L(G),x^{-1}) \\
&\quad + mx^{2m+n-3}M_0(G,x^{-1}) - x^{2m+n-3}M_1^*(G,x^{-1}) \\
M_2(T^{+--}(G),x) &= mx^{m^2} + M_{n-2,n-2}(\overline{L(G)},x) \\
&\quad + mx^{m(m+n-3)}M_0(G,x^{-m}) - x^{m(m+n-3)}M_1^*(G,x^{-m}) \\
M_3(T^{+--}(G),x) &= m + \overline{M_3}(L(G),x) + mx^{n-3}M_0(G,x) - x^{n-3}M_1^*(G,x).
\end{aligned}$$

Theorem 3.13 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{-0-}(G),x) &= x^{-2(n+m-1)}\overline{M_1}(G,x^2)+mx^{2n+m-3}M_0(G,x^{-2})\\
&\quad -x^{(2n+m-3)}M_1^*(G,x^{-2})\\
M_2(T^{-0-}(G),x) &= \overline{M'}_{-(n+m-1),-(n+m-1)}(G,x^2)+mx^{(n-2)(n+m-1)}M_0(G,x^{-2(n-2)})\\
&\quad +mx^{(n-2)(n+m-1)}M_1^*(G,x^{-2(n-2)})\\
M_3(T^{-0-}(G),x^{-2}) &= \overline{M_3}(G,x^2)+mx^{m+1}M_0(G,x^{-2})-x^{m+1}M_1^*(G,x^{-2}).
\end{aligned}$$

Theorem 3.14 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{-1-}(G),x) &= x^{2(n+m-1)}\overline{M_1}(G,x^{-2})+\binom{m}{2}x^{2(n+m-3)}+mx^{2(n+m-2)}M_0(G,x^{-2})\\
&\quad -x^{2(n+m-3)}M_1^*(G,x)\\
M_2(T^{-1-}(G),x) &= \overline{M'}_{-(n+m-1),-(n+m-1)}(G,x^2)+\binom{m}{2}x^{(n+m-3)^2}\\
&\quad +mx^{(n+m-3)(n+m)}M_0(G,x^{-2(n+m-3)})\\
&\quad -x^{(n+m-3)(n+m)}M_1^*(G,x^{-2(n+m-3)})\\
M_3(T^{-1-}(G),x^{-2}) &= \overline{M_3}(G,x^2)+\binom{m}{2}+mx^{-2}M_0(G,x^2)-x^{-2}M_1^*(G,x^2).
\end{aligned}$$

Theorem 3.15 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{-+-}(G),x) &= x^{2(n+m-1)}\overline{M_1}(G,x^{-2})+x^{(n-2)}M_1(L(G),x)+x^{2n+m-3}\sum_{u\nsim v}x^{-2d_G(u)+d_G(v)}\\
M_2(T^{-+-}(G),x) &= \overline{M'}_{-(n+m-1),-(n+m-1)}(G,x^2)+M'_{(n+2,n+2)}(L(G),x)\\
&\quad +\sum_{u\nsim v}x^{(n+m-1-2d_G(u))(n-2+d_G(v))}\\
M_3(T^{-+-}(G),x) &= x^{|m+1|}M'_{-2,-1}(\overline{G},x)+M_3(L(G),x)+x^{m+1}\sum_{u\nsim v}x^{|2d_G(u)+d_G(v)|}.
\end{aligned}$$

Theorem 3.16 *Let G be a graph of order n and size m. Then*

$$\begin{aligned}
M_1(T^{---}(G),x) &= x^{2(n+m-1)}\overline{M_1}(G,x^{-2})+x^{2(n+m-3)}\overline{M_1}(L(G),x^{-1})\\
&\quad +x^{2(n+m-1)}\sum_{u\nsim v}x^{-2d_G(u)-d_G(v)}\\
M_2(T^{---}(G),x) &= \overline{M'}_{-(n+m-1),-(n+m-1)}(G,x^2)+\overline{M'}_{-(n+m-3),-(n+m-3)}(L(G),x)\\
&\quad +\sum_{u\nsim v}x^{(n+m-1-d_G(u))(n+m-3-d_G(v))}\\
M_3(T^{---}(G),x) &= \overline{M_3}(G,x)+\overline{M_3}(L(G),x)+x^2\sum_{u\nsim v}x^{|d_G(u)-d_G(v)|}
\end{aligned}$$

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On (j, m) Symmetric Convex Harmonic Functions

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Abstract: In the present paper we define and investigate a new class of sense preserving harmonic univalent functions $HCV^{j,m}(k,\alpha)$ related to uniformly convex analytic functions. We obtain co-efficient bounds, distortion theorem and extreme points.

Key Words: (j, m) Symmetric functions, harmonic Functions, uniformly convex analytic functions.

AMS(2010): 30C45.

§1. Introduction

Let $\mathcal{U} = \{z : |z| < 1\}$ denote an open unit disc and let H denote the class of all complex valued, harmonic and sense preserving univalent functions f in $\mathcal{U}$ normalized by $f(0) = f_z(0) - 1 = 0$. Each $f \in H$ can be expressed by $f = h + \bar{g}$ where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad , \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad |b_1| < 1, \tag{1.1}$$

are analytic in $\mathcal{U}$. A necessary and sufficient condition for f to be locally univalent and sense - preserving in $\mathcal{U}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{U}$. Clunie and Sheil-Small [3] studied H together with some geometric sub-classes of H. We note that the family H of orientation preserving, normalized harmonic univalent functions reduces to the well known class S of normalized univalent functions in $\mathcal{U}$, if the co-analytic part of f is identically zero, that is $g \equiv 0$. Harmonic functions are famous for their use in the study of minimal surfaces and also play important roles in a variety of problems in applied mathematics. We can find more details in [1, 2, 4, 5]. Also let $\overline{H}$ denote the subclass of H consisting of functions $f = h + \bar{g}$ so that the functions h and g take the form

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad , \qquad g(z) = -\sum_{n=1}^{\infty} b_n z^n, \qquad |b_1| < 1. \tag{1.2}$$

Definition 1.1 *Let m be any positive integer. A domain $\mathcal{D}$ is said to be m-fold symmetric if a rotation of $\mathcal{D}$ about the origin through an angle $\frac{2\pi}{m}$ carries $\mathcal{D}$ onto itself. A function f is said*

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to be m-fold symmetric in $\mathcal{D}$ if for every z in $\mathcal{D}$ we have

$$f\left(e^{\frac{2\pi i}{m}}z\right)=e^{\frac{2\pi i}{m}}f(z),\ z\in\mathcal{D}.$$

The family of all k-fold symmetric functions is denoted by $\mathcal{S}^k$, and for $k=2$ we get class of odd univalent functions. The notion of (j,m)-symmetrical functions ($m=2,3,\cdots$, and $j=0,1,2,\cdots,m-1$) is a generalization of the notion of even, odd, k-symmetrical functions and also generalizes the well-known result that each function defined on a symmetrical subset can be uniquely expressed as the sum of an even function and an odd function. The theory of (j,m)-symmetrical functions has many interesting applications; for instance, in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan's uniqueness theorem for holomorphic mappings, see [8]. Denote the family of all (j,m)-symmetrical functions by $\mathcal{S}^{(j,m)}$. We observe that, $\mathcal{S}^{(0,2)}$, $\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,m)}$ are the classes of even, odd and m-symmetric functions respectively. We have the following decomposition theorem.

Theorem 1.2([8]) *For every mapping $f:\mathcal{U}\mapsto\mathbb{C}$, and a m-fold symmetric set, there exists exactly one sequence of (j,m)-symmetrical functions $f_{j,m}$ such that*

$$f(z)=\sum_{j=0}^{m-1}f_{j,m}(z),$$

where

$$f_{j,m}(z)=\frac{1}{m}\sum_{v=0}^{m-1}\varepsilon^{-vj}f\left(\varepsilon^{v}z\right),\ z\in\mathcal{U}. \tag{1.3}$$

Remark 1.3 Equivalently, (1.3) may be written as

$$f_{j,m}(z)=\sum_{n=1}^{\infty}\delta_{n,j}a_nz^n,\quad a_1=1, \tag{1.4}$$

where

$$\delta_{n,j}=\frac{1}{m}\sum_{v=0}^{m-1}\varepsilon^{(n-j)v}=\begin{cases}1, & n=lm+j;\\ 0, & n\neq lm+j;\end{cases}, \tag{1.5}$$

$$(l\in\mathbb{N},\ m=1,2,\cdots,\ j=0,1,2,\cdots,m-1).$$

Yong Chan Kim et al [7] discussed the class $HCV(k,\alpha)$ of complex valued, sense preserving harmonic univalent functions. f of the form (1.1) and satisfying

$$R\left\{1+(1+ke^{i\phi})\frac{z^2h''(z)+\overline{2zg'(z)+z^2g''(z)}}{zh'(z)-zg'(z)}\right\}\geq\alpha, 0\leq\alpha<1. \tag{1.6}$$

Now, using the concept of (j,m) symmetric points we define the following.

Definition 1.4 *For $0 \leq \alpha < 1$ and $m = 1, 2, 3, \cdots, j = 0, 1, 2, \cdots, m-1$. Let $HCV^{j,m}(k, \alpha)$ which denote the class of sense-preserving, harmonic univalent functions f of the form* (1.1) *which satisfy the condition*

$$\begin{aligned} \frac{\partial}{\partial\theta}\left(arg f(re^{i\theta})\right) &= Im\left(\frac{\frac{\partial}{\partial\theta} f'(re^{i\theta})}{f'_{j,m}(re^{i\theta})}\right) \\ &= Re\left\{1 + (1 + ke^{i\phi})\frac{z^2h''(z) + \overline{2zg'(z) + z^2g''(z)}}{zh'_{j,m}(z) - \overline{zg'_{j,m}(z)}}\right\} \geq \alpha. \end{aligned} \tag{1.7}$$

where $z = re^{i\theta}, 0 \leq r < 1, 0 \leq \theta < 2\pi$, $0 \leq k < \infty$ and $f_{j,m} = h_{j,m} + \overline{g_{j,m}}$ where $h_{j,m}, g_{j,m}$ given by

$$h_{j,m}(z) = \frac{1}{m}\sum_{v=0}^{m-1}\varepsilon^{-vj}h(\varepsilon^v z), g_{j,m}(z) = \frac{1}{m}\sum_{v=0}^{m-1}\varepsilon^{-vj}g(\varepsilon^v z). \tag{1.8}$$

We need the following result due to Jahangiri [6] to prove our main results.

Theorem 1.5 *Let $f = h + \bar{g}$ with h and g of the form* (1.1)*. If*

$$\sum_{n=1}^{\infty}\frac{n(n-\alpha)}{1-\alpha}|a_n| + \sum_{n=1}^{\infty}\frac{n(n+\alpha)}{1-\alpha}|b_n| \leq 2, \quad a_1 = 1,\ 0 \leq \alpha < 1, \tag{1.9}$$

then f is harmonic, sense-preserving, univalent in $\mathcal{U}$, and f is convex harmonic of order α denoted by $HK(\alpha)$. Notice that the condition (1.9) *is also necessary if $f \in \overline{H}K(\alpha) \equiv HK(\alpha) \cap \overline{H}$.*

§2. Main Results

Theorem 2.1 *Let $f = h + \overline{g}$ of the form* (**??**) *and $f_{j,m} = h_{j,m} + \overline{g}_{j,m}$ with $h_{j,m}$ and $\overline{g}_{j,m}$ given by* (1.8)*. If $0 \leq k < \infty, 0 \leq \alpha < 1, m = 1, 2, 3, \cdots, j = 0, 1, 2, \cdots, m-1$ and*

$$\sum_{n=1}^{\infty}\frac{n[n(k+1) - k - \alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})} \mid a_n \mid + \sum_{n=1}^{\infty}\frac{n[n(k+1) + k + \alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})} \mid b_n \mid \leq 2, \tag{2.1}$$

then f is harmonic, sense- preserving, univalent in $\mathcal{U}$, and $f \in HCV^{j,m}(k, \alpha)$, where $\delta_{n,j}$ given by (1.5).

Proof Since $n - \alpha \leq n + nk - k - \alpha\delta_{n,j}$ and $n + \alpha \leq n + nk + k + \alpha\delta_{n,j}$ for $0 \leq k < \infty$, it follows from Theorem 1.5 that $f \in HK(\alpha)$ and hence f is sense- preserving and convex univalent in $\mathcal{U}$. Now we need to show that if (2.1) holds then

$$\begin{aligned} &Re\left\{\frac{zh'(z) + (1 + ke^{i\phi})z^2h''(z) + (1 + 2ke^{i\phi})\overline{zg'(z)} + (1 + ke^{i\phi})\overline{z^2g''(z)}}{zh'_{j,m}(z) - \overline{zg'_{j,m}(z)}}\right\} \\ &= Re\left(\frac{A(z)}{B(z)}\right) \geq \alpha. \end{aligned} \tag{2.2}$$

Using the fact that $Re(w) \geq \alpha$ if and only if $\mid 1-\alpha+w \mid \geq \mid 1+\alpha-w \mid$ it suffices to show that

$$\mid A(z)+(1-\alpha)B(z) \mid - \mid A(z)-(1+\alpha)B(z) \mid \quad \geq 0, \tag{2.3}$$

where $A(z) = zh'(z)+(1+ke^{i\phi})z^2h''(z)+(1+2ke^{i\phi})\overline{zg'(z)}+(1+ke^{i\phi})\overline{z^2g''(z)}$ and $B(z) = zh'_{j,m}(z)-\overline{zg'_{j,m}(z)}$. substituting for A(z) and B(z) in (2.3), we obtain

$$\begin{aligned}
&|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)| \\
&= \Big|zh'(z)+(1+ke^{i\phi})z^2h''(z)+(1+2ke^{i\phi})\overline{zg'(z)}+(1+ke^{i\phi})\overline{z^2g''(z)} \\
&\qquad\qquad + (1-\alpha)[zh'_{j,m}(z)-\overline{zg'_{j,m}(z)}]\Big| \\
&\quad - \Big|zh'(z)+(1+ke^{i\phi})z^2h''(z)+(1+2ke^{i\phi})\overline{zg'(z)}+(1+ke^{i\phi})\overline{z^2g''(z)} \\
&\qquad\qquad -(1+\alpha)[zh'_{j,m}(z)-\overline{zg'_{j,m}(z)}]\Big| \\
&= \Big|[1+(1-\alpha)\delta_{1,j}]z+\sum_{n=2}^{\infty} n[n+(n-1)ke^{i\phi}+(1-\alpha)\delta_{n,j}]a_nz^n \\
&\qquad\qquad + \sum_{n=1}^{\infty} n[n+k(n+1)ke^{i\phi}-(1-\alpha)\delta_{n,j}]\ \overline{b_n}\overline{z^n}\Big| \\
&\quad - \Big|[1-(1+\alpha)\delta_{1,j}]z+\sum_{n=2}^{\infty} nn[n+(n-1)ke^{i\phi}-(1+\alpha)\delta_{n,j}]a_nz^n \\
&\qquad\qquad - \sum_{n=1}^{\infty} n[n+k(n+1)ke^{i\phi}+(1+\alpha)\delta_{n,j}]\overline{b_n}\overline{z^n}\Big| \\
&\geq [1+(1-\alpha)\delta_{1,j}] \mid z \mid - \sum_{n=2}^{\infty} n[n(k+1)-k-(1-\alpha)\delta_{n,j}] \mid a_n \mid\mid z^n \mid \\
&\qquad\qquad - \sum_{n=1}^{\infty} n[n(k+1)+k+(1-\alpha)\delta_{n,j}] \mid b_n \mid\mid z^n \mid \\
&= (2(1-\alpha)\delta_{1,j}) \mid z \mid \left\{1-\sum_{n=2}^{\infty}\frac{n[n(k+1)-k-\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})} \mid a_n \mid\mid z \mid^{n-1}\right. \\
&\qquad\qquad \left.-\sum_{n=1}^{\infty}\frac{n[n(k+1)+k+\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})} \mid b_n \mid\mid z \mid^{n-1}\right\} \\
&\geq (2(1-\alpha)\delta_{1,j}) \mid z \mid \left\{1-\left(\sum_{n=2}^{\infty}\frac{n[n(k+1)-k-\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})} \mid a_n \mid\right.\right. \\
&\qquad\qquad \left.\left.+\sum_{n=1}^{\infty}\frac{n[n(k+1)+k+\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})} \mid b_n \mid\right)\right\} \geq 0
\end{aligned}$$

by (2.1). The harmonic functions

$$f(z) = z+\sum_{n=1}^{\infty}\frac{(1-\alpha\delta_{1,j})}{n[n(k+1)-k-\alpha\delta_{n,j}]}|a_n| \ +\sum_{n=1}^{\infty}\frac{(1-\alpha\delta_{1,j})}{n[n(k+1)+k+\alpha\delta_{n,j}]}|b_n| \leq 2, \tag{2.4}$$

where

$$\sum_{n=1}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=2,$$

show that the coefficient bound given in Theorem 2.1 is sharp. The functions of the form (2.4) are in $HCV^{j,m}(k,\alpha)$ because

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{n[n(k+1)-k-\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})}\mid a_n\mid+\sum_{n=1}^{\infty}\frac{n[n(k+1)+k+\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})}\mid b_n\mid\\&=\sum_{n=1}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=2.\end{aligned}\tag{2.5}$$

This completes the proof. □

If $j=m=1$ we get the following result proved by Yong Chan Kim et al in [7].

Corollary 2.2 *Let $f=h+\overline{g}$ of the form (??). If $0\le k<\infty,\ \ 0\le\alpha<1$ and*

$$\sum_{n=1}^{\infty}\frac{n(n+nk-k-\alpha)}{(1-\alpha)}|a_n|+\sum_{n=1}^{\infty}\frac{n(n+nk+k+\alpha)}{(1-\alpha)}|b_n|\le 2,$$

then f is harmonic, sense- preserving, univalent in $\mathcal{U}$, and $f\in HCV(k,\alpha)$.

Now we show that the bound (2.1) is also necessary for functions in $\overline{H}CV(k,\alpha)$.

Theorem 2.3 *Let $f=h+\overline{g}$ of the form (1.2) and $f_{j,m}=h_{j,m}+\overline{g}_{j,m}$ with $h_{j,m}$ and $\overline{g}_{j,m}$ given by (1.8). Then $f\in\overline{H}CV^{j,m}(k,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}\frac{n[n(k+1)-k-\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})}\mid a_n\mid+\sum_{n=1}^{\infty}\frac{n[n(k+1)+k+\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})}\mid b_n\mid\le 2\tag{2.6}$$

where $0\le k<\infty, 0\le\alpha<1, m=1,2,3,\cdots,j=0,1,2,\cdots,m-1,$ and $\delta_{n,j}$ given by (1.5).

Proof In view of Theorem 2.3, we only need to show that $\overline{H}CV^{j,m}(k,\alpha)$ if condition (2.6) does not hold. We note that a necessary and sufficient condition for $f=h+\overline{g}$ of the form (1.1) to be satisfied. Equivalently, we must have

$$\begin{aligned}&Re\left\{\frac{A(z)}{B(z)}-\alpha\right\}\\&=Re\left\{\frac{zh'(z)+(1+ke^{i\phi})z^2h''(z)+(1+2ke^{i\phi})(\overline{zg'(z)})+(1+ke^{i\phi})(\overline{z^2g''(z)})}{zh'_{j,k}(z)-\overline{zg'_{j,k}(z)}}-\alpha\right\}\ge 0.\end{aligned}$$

Therefore,

$$Re\left\{\frac{(1-\delta_{1,j}\alpha)z-\sum_{n=2}^{\infty}n[n(k+1)-k-\alpha\delta_{n,j}]|a_n|z^n+\sum_{n=1}^{\infty}n[n(k+1)+k+\alpha\delta_{n,j}]\overline{|b_n|z^n}}{\delta_{1,j}z-\sum_{n=2}^{\infty}n\delta_{n,j}|a_n|z^n+\sum_{n=1}^{\infty}n\delta_{n,j}|b_n|z^n}\right\}\ge 0\tag{2.7}$$

upon choosing the value of z on the positive real axis where $0\le z=r<1$ the above inequality reduces to

$$\frac{(1-\delta_{1,j}\alpha)-\left\{\sum_{n=2}^{\infty} n[n(k+1)-k-\alpha\delta_{n,j}]|a_n|+\sum_{n=1}^{\infty} n[n(k+1)+k+\alpha\delta_{n,j}]\overline{|b_n|}\right\} r^{n-1}}{\delta_{1,j}-\sum_{n=2}^{\infty} n\delta_{n,j}|a_n|r^{n-1}+\sum_{n=1}^{\infty} n\delta_{n,j}|b_n|r^{n-1}} \geq 0. \tag{2.8}$$

If condition (2.6) does not hold then the numerator in (2.8) is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in (0,1) for which the quotient (2.8) is negative. This contradicts the required condition for $f \in \overline{H}CV^{j,m}(k,\alpha)$and so proof is complete. □

§3. Extreme Points and Distortion Bounds

Theorem 3.1 *Let f be of the form of* (1.2). *Then $f \in clco\overline{H}CV(k,\alpha)$ if and only if $f(z) = \sum_{n=1}^{\infty}(\tau_n h_n(z)+\lambda_n g_n(z))$ where $h_1(z) = z$, $h_n(z) = z - \frac{(1-\alpha\delta_{1,j})}{n[n(k+1)-k-\alpha\delta_{n,j}]}z^n, (n = 2,3,4,\cdots,)$ and $g_n(z) = z - \frac{(1-\alpha\delta_{1,j})}{n[n(k+1)+k+\alpha\delta_{n,j}]}\bar{z}^n, (n = 1,2,3,\cdots,)$, $\sum_{n=1}^{\infty}(\tau_n+\lambda_n) = 1, \tau_n \geq 0$ and $\lambda_n \geq 0$, In particular, the extreme points of $\overline{H}CV^{j,m}(k,\alpha)$ are $\{h_n\}$and$\{g_n\}$, and $\delta_{n,j}$ given by* (1.5).

Proof For functions of f of the form $f(z) = \sum_{n=1}^{\infty}(\tau_n h_n(z)+\lambda_n g_n(z))$, we have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty}(\tau_n+\lambda_n)z - \sum_{n=2}^{\infty}\frac{(1-\alpha\delta_{1,j})}{n[n(k+1)-k-\alpha\delta_{n,j}]}\tau_n z^n \\ &\quad -\sum_{n=1}^{\infty}\frac{(1-\alpha\delta_{1,j})}{n[n(k+1)+k+\alpha\delta_{n,j}]}\lambda_n z^{-n} = z - \sum_{n=2}^{\infty} a_n z^n - \sum_{n=1}^{\infty} b_n z^{-n}. \end{aligned} \tag{3.1}$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{n[n(k+1)-k-\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})}|a_n| + \sum_{n=1}^{\infty}\frac{n[n(k+1)+k+\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})}|b_n| = \sum_{n=2}^{\infty}\tau_n + \sum_{n=1}^{\infty}\lambda_n = 1-\tau_1 \leq 1$$

and so $f \in \overline{H}CV^{j,m}(k,\alpha)$.

Conversely, Suppose that $f \in \overline{H}CV^{j,m}(k,\alpha)$. We set $\tau_n = \frac{n[n(k+1)-k-\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})}|a_n|, n = 2,3,4,\cdots,$ $\lambda_n = \frac{n[n(k+1)+k+\alpha\delta_{n,j}]}{(1-\alpha\delta_{1,j})}|b_n|, n = 1,2,3,\cdots,$ and $\tau_1 = 1-\sum_{n=2}^{\infty}\tau_n - \sum_{n=1}^{\infty}\lambda_n$. Then $\sum_{n=1}^{\infty}(\tau_n+\lambda_n) = 1$, $0 \leq \tau_n \leq 1$, $0 \leq \lambda_n \leq 1, (n = 1,2,3,\cdots,)$ thus by simple calculations we get $f(z) = \sum_{n=1}^{\infty}(\tau_n h_n(z)+\lambda_n g_n(z))$ and the proof is complete. □

Theorem 3.2 *If $f \in \overline{H}CV^{j,m}(k,\alpha)$ then*

$$|f(z) \leq (1+|b_1|)r + \frac{1}{2}\left[\frac{1-\alpha\delta_{1,j}}{k+2-\alpha\delta_{2,j}} - \frac{1+2k+\alpha\delta_{1,j}}{k+2-\alpha\delta_{2,j}}|b_1|\right]r^2, \quad |z| = r < 1$$

and

$$|f(z) \geq (1-|b_1|)r - \frac{1}{2}\left[\frac{1-\alpha\delta_{1,j}}{k+2-\alpha\delta_{2,j}} - \frac{1+2k+\alpha\delta_{1,j}}{k+2-\alpha\delta_{2,j}}|b_1|\right]r^2, \quad |z| = r < 1$$

where $0 \leq \alpha < 1$, and $\delta_{n,j}$ given by (1.5).

Proof Calculation shows that

$$
\begin{aligned}
|f(z)| &\leq (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n \leq (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2 \\
&\leq (1+|b_1|)r \\
&\quad + \frac{1-\alpha\delta_{1,j}}{2[(k+2)-\alpha\delta_{2,j}]}\left\{\sum_{n=2}^{\infty}\frac{2[(k+2)-\alpha\delta_{2,j}]}{1-\alpha\delta_{1,j}}|a_n| + \sum_{n=2}^{\infty}\frac{2[(k+2)-\alpha\delta_{2,j}]}{1-\alpha\delta_{1,j}}|b_n|\right\}r^2 \\
&\leq (1+|b_1|)r + \frac{1-\alpha\delta_{1,j}}{2[(k+2)-\alpha\delta_{2,j}]}\left\{\sum_{n=2}^{\infty}\frac{n[n(k+1)-k-\alpha\delta_{n,j}]}{1-\alpha\delta_{1,j}}|a_n|\right. \\
&\qquad\qquad \left. + \sum_{n=2}^{\infty}\frac{n[n(k+1)+k+\alpha\delta_{n,j}]}{1-\alpha\delta_{1,j}}|b_n|\right\}r^2 \\
&\leq (1+|b_1|)r + \frac{1-\alpha\delta_{1,j}}{2[(k+2)-\alpha\delta_{2,j}]}\left\{1-\frac{1+2k+\alpha\delta_{1,j}}{1-\alpha\delta_{1,j}}|b_1|\right\}r^2 \\
&\leq (1+|b_1|)r + \frac{1}{2}\left[\frac{1-\alpha\delta_{1,j}}{k+2-\alpha\delta_{2,j}} - \frac{1+2k+\alpha\delta_{1,j}}{k+2-\alpha\delta_{2,j}}|b_1|\right]r^2
\end{aligned}
$$

and

$$
\begin{aligned}
|f(z)| &\geq (1+|b_1|)r - \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n \geq (1+|b_1|)r - \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2 \\
&\geq (1-|b_1|)r \\
&\quad - \frac{1-\alpha\delta_{1,j}}{2[(k+2)-\alpha\delta_{2,j}]}\left\{\sum_{n=2}^{\infty}\frac{2[(k+2)-\alpha\delta_{2,j}]}{1-\alpha\delta_{1,j}}|a_n| + \sum_{n=2}^{\infty}\frac{2[(k+2)-\alpha\delta_{2,j}]}{1-\alpha\delta_{1,j}}|b_n|\right\}r^2 \\
&\geq (1-|b_1|)r - \frac{1-\alpha\delta_{1,j}}{2[(k+2)-\alpha\delta_{1,j}]}\left\{\sum_{n=2}^{\infty}\frac{n[n(k+1)-k-\alpha\delta_{n,j}]}{1-\alpha\delta_{1,j}}|a_n|\right. \\
&\qquad\qquad \left. + \sum_{n=2}^{\infty}\frac{n[n(k+1)+k+\alpha\delta_{n,j}]}{1-\alpha\delta_{1,j}}|b_n|\right\}r^2 \\
&\geq (1-|b_1|)r - \frac{1-\alpha\delta_{1,j}}{2[(k+2)-\alpha\delta_{2,j}]}\left\{1-\frac{1+2k+\alpha\delta_{1,j}}{1-\alpha\delta_{1,j}}|b_1|\right\}|b_1|r^2 \\
&\geq (1-|b_1|)r - \frac{1}{2}\left[\frac{1-\alpha\delta_{2,j}}{k+2-\alpha\delta_{2,j}} - \frac{1+2k+\alpha\delta_{1,j}}{k+2-\alpha\delta_{2,j}}|b_1|\right]r^2.
\end{aligned}
$$

This completes the proof. □

If $j=m=1$ we get the following result proved by Yong Chan Kim et al. in [7]

Corollary 3.3 *If $f \in \overline{H}CV(k,\alpha)$ then*

$$|f(z) \leq (1+|b_1|)r + \frac{1}{2}\left[\frac{1-\alpha}{k+2-\alpha} - \frac{1+2k+\alpha}{k+2-\alpha}|b_1|\right]r^2, \quad |z|=r<1$$

and

$$|f(z) \geq (1-|b_1|)r - \frac{1}{2}\left[\frac{1-\alpha}{k+2-\alpha} - \frac{1+2k+\alpha}{k+2-\alpha}|b_1|\right]r^2, \quad |z|=r<1.$$

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On M-Projective Curvature Tensor of a $(LCS)_n$-Manifold

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Abstract: In the present paper, we characterize the M-projective curvature tensor in a $(LCS)_n$-manifold. Geometric properties on the curvature tensor such as ϕ-M-projective flat, M-projective pseudosymmetric, ϕ-M-projective semisymmetric and generalized M-projective ϕ-recurrent are studied on $(LCS)_n$-manifold.

Key Words: $(LCS)_n$-Manifold, M-projective curvature tensor, ϕ-M-projective semisymmetric, ϕ-M-projective flat, generalized M-projective ϕ-recurrent, M-projective pseudosymmetric.

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§1. Introduction

In 2003, Shaikh [18] introduced and studied Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds) with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [11]. Also Shaikh et al. ([19,20,21,22]), Prakasha [16], Yadav [29] studied various types of $(LCS)_n$-manifolds by imposing curvature restrictions.

In 1926, the concept of local symmetry of a Riemannian manifold was started by Cartan [3]. This notion has been used in several directions by many authors such as recurrent manifolds by Walker [28], semi-symmetric manifold by Szabo [24], pseudosymmetric manifold by Chaki [4], pseudosymmetric spaces by Deszcz [10], weakly symmetric manifold by Tamassy and Binh [26], weakly symmetric Riemannian spaces by Selberg [17]. The notions of pseudo-symmetric and weak symmetry by Chaki and Deszcz and Selberge and Tamassy and Binh respectively are different. As a mild version of local symmetry, Takahashi [25] introduced the notion of ϕ-symmetry on a Sasakian manifold. In 2003, De et al. [7] introduced the concept of ϕ-recurrent Sasakian manifold, which generalizes the notion of ϕ-symmetry.

In 1971, Pokhariyal and Mishra [15] defined a tensor field W^* on a Riemannian manifold given by

$$\begin{aligned} W^*(X,Y)Z &= R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad -g(X,Z)QY]. \end{aligned} \tag{1.1}$$

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Such a tensor field W^* is known as M-projective curvature tensor. Ojha [13,14] studied M-projective curvature tensor on Sasakian and Kaehler manifold. The properties of M-projective curvature tensor were also studied on different manifolds by Chaubey [5,6], Venkatesha [27] and others.

Motivated by the above studies, we made an attempt to study M-projective curvature tensor on $(LCS)_n$-manifold.

The present paper is organized as follows: Section 2 is equipped with some preliminaries of $(LCS)_n$ manifold. In section 3, we proved that if (M^n, g) is an n-dimensional ϕ-M-projective flat $(LCS)_n$-manifold, then the manifold M^n is η-Einstein manifold. We have shown that if an n-dimensional $(LCS)_n$-manifold M^n is M-projective pseudosymmetric then either $L_{W^*} = (\alpha^2 - \rho)$ or the manifold is Einstein manifold, provided $(\alpha^2 - \rho) \neq 0$, in section 4. Section 5 deals with the study of ϕ-M-projective semisymmetric $(LCS)_n$-manifold and proved that the manifold is generalized η-Einstein manifold, provided $(\alpha^2 - \rho) \neq 0$. In the last section, we have studied generalized M-projective ϕ-recurrent $(LCS)_n$-manifold and gave the relations between the associated 1-forms A and B.

§2. Preliminaries

An n-dimensional Lorentzian manifold M^n is a smooth connected para-compact Hausdorff manifold with a Lorentzian metric g of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p : T_p(M^n) \times T_p(M^n) \to R$ is a non-degenerate inner product of signature $(-, +, +, \cdots, +)$, where $T_p(M^n)$ denotes the tangent space of M^n at p and R is the real number space [18,12].

In a Lorentzian manifold (M^n, g), a vector field P defined by

$$g(X, P) = A(X),$$

for any vector field $X \in \chi(M^n)$, ($\chi(M^n)$ being the Lie algebra of vector fields on M^n) is said to be a concircular vector field [23] if

$$(\nabla_X A)(Y) = \alpha[g(X, Y) + \omega(X)A(Y)],$$

where α is a non-zero scalar function, A is a 1-form and ω is a closed 1-form.

Let M^n be a Lorentzian manifold admitting a unit time like concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1. \tag{2.1}$$

Since ξ is a unit concircular vector field, there exists a non-zero 1-form η such that for

$$g(X, \xi) = \eta(X), \tag{2.2}$$

the equation of the following form holds

$$(\nabla_X \eta)(Y) = \alpha[g(X,Y) + \eta(X)\eta(Y)], \quad (\alpha \neq 0) \tag{2.3}$$

for all vector fields X and Y. Here ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non zero scalar function satisfying

$$(\nabla_X \alpha) = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.4}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{2.5}$$

then from (2.3) and (2.5) we have

$$\phi^2 X = X + \eta(X)\xi, \tag{2.6}$$

$$\eta(\xi) = -1, \ \ \phi\xi = 0, \ \ \eta(\phi X) = 0, \ \ g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.7}$$

from which it follows that ϕ is a symmetric $(1,1)$ tensor, called the structure tensor of the manifold. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated 1-form η and $(1,1)$ tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n$-manifold) [18]. Especially, if we take $\alpha = 1$, then we obtain the LP-Sasakian structure of Matsumoto [11].

In a $(LCS)_n$-manifold, the following relations hold [18]:

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.8}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.9}$$

$$R(X,\xi)Z = (\alpha^2 - \rho)[\eta(Z)X - g(X,Z)\xi], \tag{2.10}$$

$$R(\xi,X)Y = (\alpha^2 - \rho)[g(X,Y)\xi - \eta(Y)X], \tag{2.11}$$

$$R(\xi,X)\xi = (\alpha^2 - \rho)[X + \eta(X)\xi], \tag{2.12}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X), \quad Q\xi = (n-1)(\alpha^2 - \rho)\xi, \tag{2.13}$$

$$(\nabla_X \phi)(Y) = \alpha[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \tag{2.14}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)(\alpha^2 - \rho)\eta(X)\eta(Y) \tag{2.15}$$

for all vector fields X, Y, Z and R, S respectively denotes the curvature tensor and the Ricci tensor of the manifold.

A $(LCS)_n$ manifold M^n is said to be a generalized η-Einstein manifold [30] if the following condition

$$S(X,Y) = \lambda g(X,Y) + \mu\eta(X)\eta(Y) + \nu\Omega(X,Y) \tag{2.16}$$

holds on M^n. Here λ, μ and ν are smooth functions and $\Omega(X,Y) = g(\phi X, Y)$. If $\nu = 0$ then the manifold reduces to an η-Einstein manifold.

From (1.1), we have

$$\eta(W^*(\xi,Y)Z) = \frac{1}{2(n-1)}S(Y,Z) - \frac{1}{2}g(Y,Z), \tag{2.17}$$

$$\eta(W^*(X,\xi)Z) = -\frac{1}{2(n-1)}S(X,Z) - \frac{3}{2}(\alpha^2-\rho)g(X,Z), \tag{2.18}$$

$$\eta(W^*(X,Y)\xi) = 0, \tag{2.19}$$

$$W^*(X,Y)\xi = 0, \quad W^*(X,\xi)\xi = 0, \quad W^*(\xi,\xi)Z = 0, \tag{2.20}$$

$$\begin{aligned} W^*(X,\xi,Z,T) = \; & \frac{1}{2(n-1)}S(X,Z)\eta(T) - \frac{1}{2(n-1)}S(X,T)\eta(Z) \\ & + \frac{1}{2}(\alpha^2-\rho)g(X,T)\eta(Z) - \frac{1}{2}(\alpha^2-\rho)g(X,Z)\eta(T), \end{aligned} \tag{2.21}$$

$$W^*(X,\xi,Z,\xi) = -\frac{1}{2(n-1)}S(X,Z) + \frac{1}{2}(\alpha^2-\rho)g(X,Z), \tag{2.22}$$

$$W^*(X,\xi)Z = \frac{1}{2(n-1)}S(X,Z)\xi - \frac{1}{2}(\alpha^2-\rho)g(X,Z)\xi, \tag{2.23}$$

$$(\nabla_U S)(X,\xi) = (n-1)\alpha(\alpha^2-\rho)[g(U,X) + \eta(U)\eta(X)] - \alpha S(X,\phi U). \tag{2.24}$$

§3. ϕ-M-projectively Flat $(LCS)_n$-Manifold

Definition 3.1 *An n-dimensional $(LCS)_n$-manifold (M^n, g), $(n > 3)$ is called ϕ-M-projective flat if it satisfies the condition*

$$\phi^2 W^*(\phi X, \phi Y)\phi Z = 0, \tag{3.1}$$

for all vector fields X, Y, Z on the manifold.

Theorem 3.1 *If (M^n, g) is an n-dimensional ϕ-M-projective flat $(LCS)_n$-manifold, then the manifold M^n is η-Einstein manifold.*

Proof Let M^n be ϕ-M-projective flat. It is easy to define that $\phi^2(W^*(\phi X, \phi Y)\phi Z) = 0$ holds if and only if

$$g(W^*(\phi X, \phi Y)\phi Z, \phi U) = 0, \tag{3.2}$$

for any vector fields $X, Y, Z, U \in TM^n$.

By virtue of (1.1) and (3.2), one can obtain

$$\begin{aligned} g(R(\phi X, \phi Y)\phi Z, \phi U) \;\; = \;\; & \frac{1}{2(n-1)}[S(\phi Y, \phi Z)g(\phi X, \phi U) - S(\phi X, \phi Z)g(\phi Y, \phi U) \\ & + g(\phi Y, \phi Z)S(\phi X, \phi U) - g(\phi X, \phi Z)S(\phi Y, \phi U)]. \end{aligned} \tag{3.3}$$

Let $\{e_1, e_2, \cdots, e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M^n. By using the fact that $\{\phi e_1, \phi e_2, \cdots, \phi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = U = e_i$ in (3.3) and sum up with respect to i, we get

$$\begin{aligned}\sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) &= \frac{1}{2(n-1)}\sum_{i=1}^{n-1}[S(\phi Y, \phi Z)g(\phi e_i, \phi e_i) \\ &\quad -S(\phi e_i, \phi Z)g(\phi Y, \phi e_i) + g(\phi Y, \phi Z)S(\phi e_i, \phi e_i) \\ &\quad -g(\phi e_i, \phi Z)S(\phi Y, \phi e_i)]. \qquad (3.4)\end{aligned}$$

It can be easily verify by a straight forward calculation that [1],

$$\sum_{i=1}^{n-1} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z), \qquad (3.5)$$

$$\sum_{i=1}^{n-1} S(\phi e_i, \phi e_i) = r - (n-1)(\alpha^2 - \rho), \qquad (3.6)$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z)S(\phi Y, \phi e_i) = S(\phi Y, \phi Z), \qquad (3.7)$$

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = (n-1) \qquad (3.8)$$

and

$$\sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = g(\phi Y, \phi Z). \qquad (3.9)$$

By virtue of (3.5) - (3.8), the equation (3.4) becomes

$$S(\phi Y, \phi Z) = \{\frac{r - (n-1)(\alpha^2 - \rho) - 2(n-1)}{n+1}\}g(\phi Y, \phi Z). \qquad (3.10)$$

On substituting (2.7) and (2.15), (3.10) yields

$$S(Y, Z) = k_1 g(Y, Z) + k_2 \eta(Y)\eta(Z), \qquad (3.11)$$

where $k_1 = \{\frac{r-(n-1)(\alpha^2-\rho)-2(n-1)}{n+1}\}$ and $k_2 = \{\frac{r-2(n-1)-(n-1)(n+1)(\alpha^2-\rho)}{n+1}\}$. Thus we proved the theorem. □

§4. M-Projective Pseudosymmetric $(LCS)_n$-Manifold

Definition 4.1 *An $(LCS)_n$-manifold (M^n, g) $(n > 3)$ is said to be M-projective pseudosymmetric if it satisfies*

$$(R(X, Y) \cdot W^*)(U, V)E = L_{W^*}((X \wedge Y) \cdot W^*)(U, V)E, \qquad (4.1)$$

for any vector fields $X, Y, U, V, E \in TM^n$.

Theorem 4.2 *If an n-dimensional $(LCS)_n$-manifold M^n is M-projective pseudosymmetric then either $L_{W^*} = (\alpha^2 - \rho)$ or the manifold is Einstein manifold, provided $(\alpha^2 - \rho) \neq 0$.*

Proof Let M^n be M-projective pseudosymmetric. Putting $Y = \xi$ in (4.1), we get

$$\begin{aligned}(R(X,\xi)\cdot W^*)(U,V)E &= L_{W^*}[(X\wedge\xi)W^*(U,V)E - W^*((X\wedge\xi)U,V)E \\ &\quad -W^*(U,(X\wedge\xi)V)E - W^*(U,V)(X\wedge\xi)E]. \end{aligned} \tag{4.2}$$

Now the left hand side of (4.2) reduces to

$$R(X,\xi)\cdot W^*(U,V)E - W^*(R(X,\xi)U,V)E - W^*(U,R(X,\xi)V)E - W^*(U,V)R(X,\xi)E. \tag{4.3}$$

In view of (2.10), (4.3) becomes

$$\begin{aligned}&(\alpha^2-\rho)[W^*(U,V,E,\xi)X - W^*(U,V,E,X)\xi - \eta(U)W^*(X,V)E + g(X,U)W^*(\xi,V)E \\ &-\eta(V)W^*(U,X)E + g(X,V)W^*(U,\xi)E - \eta(T)W^*(U,V)X + g(X,E)W^*(U,V)\xi].\end{aligned} \tag{4.4}$$

Similarly, right hand side of (4.2) reduces to

$$\begin{aligned}&L_{W^*}[W^*(U,V,E,\xi)X - W^*(U,V,E,X)\xi - \eta(U)W^*(X,V)E + g(X,U)W^*(\xi,V)E \\ &-\eta(V)W^*(U,X)E + g(X,V)W^*(U,\xi)E - \eta(E)W^*(U,V)X + g(X,E)W^*(U,V)\xi].\end{aligned} \tag{4.5}$$

On replacing the expressions (4.4) and (4.5) in (4.2), we get

$$\begin{aligned}&[L_{W^*} - (\alpha^2-\rho)]\{W^*(U,V,E,\xi)X - W^*(U,V,E,X)\xi - \eta(U)W^*(X,V)E \\ &+g(X,U)W^*(\xi,V)E - \eta(V)W^*(U,X)E + g(X,V)W^*(U,\xi)E - \eta(E)W^*(U,V)X \\ &+g(X,E)W^*(U,V)\xi\} = 0.\end{aligned} \tag{4.6}$$

Taking $V = \xi$ and using (2.2) and (2.7) in the above equation, we obtain

$$\begin{aligned}&[L_{W^*} - (\alpha^2-\rho)]\{W^*(U,\xi,E,\xi)X - W^*(U,\xi,E,X)\xi - \eta(U)W^*(X,\xi)E \\ &+g(X,U)W^*(\xi,\xi)E + W^*(U,X)E + \eta(X)W^*(U,\xi)E - \eta(E)W^*(U,\xi)X \\ &+g(X,E)W^*(U,\xi)\xi\} = 0.\end{aligned} \tag{4.7}$$

On using (2.21) - (2.23), (4.7) gives either $L_{W^*} = (\alpha^2 - \rho)$ or

$$\begin{aligned}W^*(U,X)E &= \frac{1}{2(n-1)}S(U,E)X + \frac{1}{2(n-1)}S(X,E)\eta(U)\xi \\ &\quad -\frac{1}{2}(\alpha^2-\rho)g(U,E)X - \frac{1}{2}(\alpha^2-\rho)g(X,E)\eta(U)\xi.\end{aligned} \tag{4.8}$$

The above equation implies

$$\begin{aligned} W^*(U,X,E,G) &= \frac{1}{2(n-1)}S(U,E)g(X,G)+\frac{1}{2(n-1)}S(X,E)\eta(U)\eta(G) \\ &\quad -\frac{1}{2}(\alpha^2-\rho)g(U,E)g(X,G)-\frac{1}{2}(\alpha^2-\rho)g(X,E)\eta(U)\eta(G). \end{aligned} \tag{4.9}$$

Contracting (4.9) gives

$$W^*(e_i,X,E,e_i)=0. \tag{4.10}$$

Simplifying (4.10), we finally obtain

$$S(X,E)=(n-1)(\alpha^2-\rho)g(X,E). \tag{4.11}$$

Thus the proof of the theorem is completed. □

§5. ϕ-M-Projectively Semisymmetric $(LCS)_n$-Manifold

Definition 5.1 *An n-dimensional $(n>3)$ $(LCS)_n$-manifold is said to be ϕ-M-projective semisymmetric if it satisfies the condition*

$$W^*(X,Y)\cdot\phi=0, \tag{5.1}$$

which turns into

$$(W^*(X,Y)\cdot\phi)Z=W^*(X,Y)\phi Z-\phi W^*(X,Y)Z=0. \tag{5.2}$$

Before we state our theorem we need the following lemma which was proved in [19].

Lemma 5.2([19]) *If M^n is an $(LCS)_n$-manifold, then for any X,Y,Z on M^n, the following relation holds:*

$$\begin{aligned} R(X,Y)\phi Z-\phi R(X,Y)Z &= (\alpha^2-\rho)[\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\xi \\ &\quad +\eta(Z)\{\eta(X)Y-\eta(Y)X\}]. \end{aligned} \tag{5.3}$$

Theorem 5.3 *If an n-dimensional $(LCS)_n$-manifold is ϕ-M-projective semisymmetric then it is a generalized η-Einstein manifold, provided $(\alpha^2-\rho)\neq 0$.*

Proof By virtue of (1.1), we have

$$\begin{aligned} W^*(X,Y)\phi Z-\phi W^*(X,Y)Z &= R(X,Y)\phi Z-\phi R(X,Y)Z-\frac{1}{2(n-1)}[S(Y,\phi Z)X \\ &\quad -S(X,\phi Z)Y+g(Y,\phi Z)QX-g(X,\phi Z)QY \\ &\quad +S(Y,Z)\phi X-S(X,Z)\phi Y+g(Y,Z)\phi QX-g(X,Z)\phi QY]. \end{aligned} \tag{5.4}$$

On using (2.13) and (5.3) in (5.4), we obtain

$$\begin{aligned}
&(\alpha^2-\rho)[\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\xi+\eta(Z)\{\eta(X)Y-\eta(Y)X\}]\\
&-\frac{1}{2(n-1)}[S(Y,\phi Z)X-S(X,\phi Z)Y+(n-1)(\alpha^2-\rho)g(Y,\phi Z)X\\
&-(n-1)(\alpha^2-\rho)g(X,\phi Z)Y+S(Y,Z)\phi X-S(X,Z)\phi Y\\
&+(n-1)(\alpha^2-\rho)g(Y,Z)\phi X-(n-1)(\alpha^2-\rho)g(X,Z)\phi Y]=0. \qquad (5.5)
\end{aligned}$$

Taking inner product of (5.5) with T, we get

$$\begin{aligned}
&(\alpha^2-\rho)[\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}\eta(T)+\eta(Z)\{\eta(X)g(Y,T)-\eta(Y)g(X,T)\}]\\
&-\frac{1}{2(n-1)}[S(Y,\phi Z)g(X,T)-S(X,\phi Z)g(Y,T)+(n-1)(\alpha^2-\rho)g(Y,\phi Z)g(X,T)\\
&-(n-1)(\alpha^2-\rho)g(X,\phi Z)g(Y,T)+S(Y,Z)g(\phi X,T)-S(X,Z)g(\phi Y,T)\\
&+(n-1)(\alpha^2-\rho)g(Y,Z)g(\phi X,T)-(n-1)(\alpha^2-\rho)g(X,Z)g(\phi Y,T)]=0. \qquad (5.6)
\end{aligned}$$

Contracting (5.6) gives

$$\begin{aligned}
S(X,\phi Z) \quad = \quad &\frac{(n-1)(n-2)}{(2-n)}(\alpha^2-\rho)g(X,\phi Z)+\frac{2(n-1)}{(2-n)}(\alpha^2-\rho)g(X,Z)\\
&+\frac{2n(n-1)}{(2-n)}(\alpha^2-\rho)\eta(X)\eta(Z). \qquad (5.7)
\end{aligned}$$

Replacing X by ϕX in the above equation, we finally obtain

$$S(X,Z)=\lambda g(X,Z)+\mu\eta(X)\eta(Z)+\nu g(\phi X,Z), \qquad (5.8)$$

where $\lambda=-(n-1)(\alpha^2-\rho)$, $\mu=-2(n-1)(\alpha^2-\rho)$ and $\nu=-\frac{2(n-1)}{(n-2)}$.

This completes the proof. □

§6. Generalized M-Projective ϕ-Recurrent $(LCS)_n$-Manifold

In 2008, Basari and Murathan [2] introduced generalized ϕ-recurrent Kenmotsu manifold. Later, De [8] and Pal [9] studied generalized concircularly recurrent and generalized M-projectively recurrent Riemannian manifold.

Definition 6.1 *A $(LCS)_n$-manifold M^n $(n>3)$ is said to be generalized M-projective ϕ-recurrent if it satisfies*

$$\phi^2((\nabla_U W^*)(X,Y,Z))=A(U)W^*(X,Y)Z+B(U)[g(Y,Z)X-g(X,Z)Y], \qquad (6.1)$$

where A and B are two 1-forms, B is non-zero and are defined by $A(U)=g(U,\rho_1)$ and $B(U)=g(U,\rho_2)$. Here ρ_1 and ρ_2 are vector fields associated to the 1-forms A and B respectively.

Theorem 6.2 *If the $(LCS)_n$-manifold M^n is generalized M-projective ϕ-recurrent, then the associated 1-forms A and B are related as follows;*

$$[\frac{n}{2}(\alpha^2-\rho)+r]A(U)+(n-1)B(U)+\frac{1}{2(n-1)}dr(U)=0. \tag{6.2}$$

Proof Suppose that M^n is generalized M-projective ϕ-recurrent $(LCS)_n$-manifold. Then, by using (2.6), (6.1) takes the form

$$(\nabla_U W^*)(X,Y)Z+\eta((\nabla_U W^*)(X,Y)Z)\xi = A(U)W^*(X,Y)Z+B(U)[g(Y,Z)X-g(X,Z)Y]. \tag{6.3}$$

From (6.3), it follows that

$$\begin{aligned}
&g((\nabla_U W^*)(X,Y,Z),T)+g((\nabla_U W^*)(X,Y,Z),\xi)g(T,\xi)\\
&-\frac{1}{2(n-1)}[(\nabla_U S)(Y,Z)g(X,T)-(\nabla_U S)(X,Z)g(Y,T)+g(Y,Z)(\nabla_U S)(X,T)\\
&-g(X,Z)(\nabla_U S)(Y,T)]-\frac{1}{2(n-1)}[(\nabla_U S)(Y,Z)\eta(X)-(\nabla_U S)(X,Z)\eta(Y)\\
&+g(Y,Z)(\nabla_U S)(X,\xi)-g(X,Z)(\nabla_U S)(Y,\xi)]\eta(T)=A(U)[g(R(X,Y)Z,T)\\
&-\frac{1}{2(n-1)}\{S(Y,Z)g(X,T)-S(X,Z)g(Y,T)+g(Y,Z)S(X,T)-g(X,Z)S(Y,T)\}]\\
&+B(U)[g(Y,Z)g(X,T)-g(X,Z)g(Y,T)].
\end{aligned} \tag{6.4}$$

On contraction, the above equation yields

$$\begin{aligned}
&(\nabla_U S)(Y,Z)+\eta((\nabla_U R)(\xi,Y,Z))-\frac{1}{2(n-1)}[(n-2)(\nabla_U S)(Y,Z)\\
&+dr(U)g(Y,Z)]-\frac{1}{2(n-1)}[-(\nabla_U S)(Y,Z)-(\nabla_U S)(Z,\xi)\eta(Y)+g(Y,Z)(\nabla_U S)(\xi,\xi)\\
&-(\nabla_U S)(Y,\xi)\eta(Z)]=A(U)[S(Y,Z)-\frac{1}{2(n-1)}\{(n-2)S(Y,Z)+rg(Y,Z)\}]\\
&+(n-1)B(U)g(Y,Z).
\end{aligned} \tag{6.5}$$

In (6.5), setting $Z=\xi$ and then using (2.2), (2.3), (2.7), (2.12) and (2.13) one can get

$$\begin{aligned}
&(\nabla_U S)(Y,\xi)[1-\frac{n-2}{2(n-1)}]-\frac{dr(U)}{2(n-1)}\eta(U)=A(U)[S(Y,\xi)\\
&-\frac{1}{2(n-1)}\{(n-2)S(Y,\xi)+r\eta(Y)\}]+(n-1)B(U)\eta(Y).
\end{aligned} \tag{6.6}$$

Now, taking $Y=\xi$ in (6.6) and using (2.2), (2.7) and (2.24), we finally obtain (6.2). □

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D-homothetic Deformations of Lorentzian Para-Sasakian Manifold

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Abstract: The aim of the present paper is to prove some results on the properties of LP-Sasakian manifolds under D-homothetic deformations. In the later sections we give several results on some properties which are conformal under the mentioned deformations. Lastly, we illustrate the main theorem by giving a detailed example.

Key Words: D-homothetic deformation, LP-Sasakian manifold, ϕ-section, sectional curvature.

AMS(2010): 53C15, 53C25.

§1. Introduction

The notion of Lorentzian almost para-contact manifolds was introduced by K. Matsumoto [3]. Later on, a large number of geometers studied Lorentzian almost para-contact manifold and their different classes, viz., Lorentzian para-Sasakian manifolds and Lorentzian special para-Sasakian manifolds [4], [5], [6], [7]. In brief, Lorentzian para-Sasakian manifolds are called LP-Sasakian manifolds. The study of LP-Sasakian manifolds has vast applications in the theory of relativity.

In an n-dimensional differentiable manifold M, (ϕ, ξ, η) is said to be an almost paracontact structure if it admits a $(1,1)$ tensor field ϕ , a timelike contravariant vector field ξ and a 1-form η which satisfy the relations:

$$\eta(\xi) = -1, \tag{1.1}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{1.2}$$

for any vector field X on M. In an n-dimensional almost paracontact manifold with structure (ϕ, ξ, η), the following conditions hold:

$$\phi\xi = 0, \tag{1.3}$$

$$\eta \circ \phi = 0, \tag{1.4}$$

$$rank\ \phi = n - 1. \tag{1.5}$$

Let M^n be differentiable manifold with an almost paracontact structure (ϕ, ξ, η). If there exists a Lorentzian metric which makes ξ a timelike unit vector field, then there exists a

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Lorentzian metric g satisfying

$$g(X,\xi) = \eta(X), \tag{1.6}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{1.7}$$

$$(\nabla_X \phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{1.8}$$

for all vector fields X, Y on $\tilde{M}$ [2].

If a differentiable manifold M admits the structure (ϕ, ξ, η, g) such that g is an associated Lorentzian metric of the almost paracontact structure (ϕ, ξ, η, g) then we say that M^n has a Lorentzian almost paracontact structure (ϕ, ξ, η, g) and M^n is said to be Lorentzian almost paracontact manifold (LAP) with structure (ϕ, ξ, η, g) .

In a LAP-manifold with structure (ϕ, ξ, η, g) if we put

$$\Omega(X,Y) = g(\phi X, Y), \tag{1.9}$$

then the tensor field Ω is a symmetric $(0,2)$ tensor field [**?**], that is

$$\Omega(X,Y) = \Omega(Y,X), \tag{1.10}$$

for all vector fields X, Y on M^n. A LAP-manifold with structure (ϕ, ξ, η, g) is said to be Lorentzian paracontact manifold if it satisfies

$$\Omega(X,Y) = \frac{1}{2}\{(\nabla_X \eta)Y + (\nabla_Y \eta)X\} \tag{1.11}$$

and (ϕ, ξ, η, g) is said to be Lorentzian paracontact structure. Here ∇ denotes the operator of covariant differentiation w.r.t the Lorentzian metric g.

In a LP-Sasakian manifold we have the following results from [9]:

$$\nabla_X \xi = \phi X, \tag{1.12}$$

$$(\nabla_X \eta)Y = \Omega(X,Y) = g(\phi X, Y), \tag{1.13}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{1.14}$$

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{1.15}$$

$$R(\xi,X)\xi = \eta(X)\xi - \eta(\xi)X = X + \eta(X)\xi, \tag{1.16}$$

$$S(X,\xi) = (n-1)\eta(X), \tag{1.17}$$

$$S(\xi,\xi) = -(n-1), \tag{1.18}$$

$$Q\xi = -(n-1), \tag{1.19}$$

where R is the curvature tensor of manifold of type $(1,3)$, S is Ricci tensor of type $(0,2)$ and Q being the Ricci operator. An example of a five-dimensional Lorentzian para-Sasakian manifold has been given by Matsumoto, Mihai and Rosaca in [5].

§2. D-homothetic Deformations of LP-Sasakian Manifolds

Let $M(\phi,\xi,\eta,g)$ be an Lorentzian almost paracontact structure. By D-homothetic deformation [8], we mean a change of structure tensors of the form

$$\overline{\eta}=a\eta,\ \overline{\xi}=\frac{1}{a}\xi,\ \overline{\phi}=\phi,\ \overline{g}=ag+a(a-1)\eta\otimes\eta,$$

where a is a positive constant.

Theorem 2.1 *Under D-homothetic deformation $M(\overline{\phi},\overline{\xi},\overline{\eta},\overline{g})$ is also an LP-Sasakian manifold $M(\phi,\xi,\eta,g)$.*

Proof Calculation shows that

$$\begin{aligned}
\overline{\eta}(\overline{\xi}) &= \overline{\eta}(\frac{1}{a}\xi)=a\eta(\frac{1}{a}\xi)=\eta(\xi)=-1,\\
\overline{\phi}^2(X) &= \phi^2(X)=X+\eta(X)\xi,\\
\overline{\phi}\circ\overline{\xi} &= \overline{\phi}(\frac{1}{a}\xi)=\phi(\frac{1}{a}\xi)=\frac{1}{a}\phi\xi=0,\\
\overline{\eta}\circ\overline{\phi} &= \overline{\eta}(\phi(X))=a\eta(\phi(X))=0,\\
rank\ \overline{\phi} &= rank\ \phi=n-1,\\
\overline{\eta}(X) &= a\eta(X)=ag(X,\xi),\\
\overline{g}(\overline{\phi}X,\overline{\phi}Y) &= \overline{g}(\phi X,\phi Y)=(ag+a(a-1)\eta\otimes\eta)(\phi X,\phi Y)=ag(\phi X,\phi Y),\\
(\nabla_X\overline{\phi})Y=(\nabla_X\phi)Y &= g(X,Y)\xi+\epsilon\eta(Y)X+2\eta(X)\eta(Y)\xi. \qquad\square
\end{aligned}$$

Theorem 2.2 *Under D-homothetic deformation of a LP Sasakian manifold the following relation holds*

$$(L_{\overline{\xi}}\overline{g})(X,Y)=a(L_\xi g)(X,Y),$$

where L_ξ is the Lie derivative.

Proof For an LP-Sasakian manifold we know $(L_\mu g)(X,Y)=2g(\phi X,Y)$ since $g(\phi X,Y)=g(X,\phi Y)$. Under D-homothetic deformation

$$\begin{aligned}
(L_{\overline{\xi}}\overline{g})(X,Y) &= 2\overline{g}(\overline{\phi}X,Y)\\
&= a(L_\xi g)(X,Y)+2(a^2-a)\eta(\phi X)\eta(Y)\\
&= a(L_\xi g)(X,Y). \qquad\square
\end{aligned}$$

§3. D-homothetic Deformations of Curvature Tensors on LP-Sasakian Manifolds

In this section we consider conformally flat LP-Sasakian manifold $M^n(\phi,\xi,\eta,g)$ $(n>3)$. The

Weyl conformal curvature tensor C is given by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad -g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{3.1}$$

For conformally flat manifold we have $C(X,Y)Z = 0$. So from (3.1) we have

$$\begin{aligned} R(X,Y)Z &= \frac{1}{n-2}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \\ &\quad -\frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{3.2}$$

Putting $Z = \xi$ in (3.2), we obtain from (1.14)

$$\begin{aligned} \eta(Y)X - \eta(X)Y &= \frac{1}{n-2}\{S(Y,\xi)X - S(X,\xi)Y + S(Y,\xi)QX - g(X,\xi)QY\} \\ &\quad -\frac{r}{(n-1)(n-2)}\{g(Y,\xi)X - g(X,\xi)Y\}. \end{aligned} \tag{3.3}$$

Putting $Y = \xi$ in (3.3) we calculate

$$\begin{aligned} \eta(\xi)X - \eta(X)\xi &= \frac{1}{n-2}\{S(\xi,\xi)X - S(X,\xi)\xi + S(\xi,\xi)QX - g(X,\xi)Q\xi\} \\ &\quad -\frac{r}{(n-1)(n-2)}\{g(\xi,\xi)X - g(X,\xi)\xi\}. \end{aligned} \tag{3.4}$$

After some steps of calculations we obtain

$$QX = (-1 + \frac{r}{n-1})X + (-1 + \frac{r}{n-1})\eta(X)\xi - (n-1)\eta(X). \tag{3.5}$$

Taking inner product with Y, above equation can be written as

$$S(X,Y) = (1 + \frac{r}{n-1})g(X,Y) + (-1 + \frac{r}{n-1})\eta(X)g(Y,\xi) - (n-1)\eta(X). \tag{3.6}$$

In view of $(3.5), (3.6)$ equation (3.2) takes the form

$$\begin{aligned} R(X,Y)Z &= [g(Y,Z)X - g(X,Z)Y]\left[\left(-1 + \frac{r}{n-1}\right)\frac{1}{n-2}\right. \\ &\quad \left. + \frac{1}{n-2}\left(1 + \frac{r}{n-1}\right) - \frac{r}{(n-1)(n-2)}\right] \\ &\quad + g(Y,Z)\eta(X)\left[(\frac{r}{n-1} - 1)\frac{1}{n-2}\xi - (n-1)\right] + g(X,Z)\eta(Y) \\ &\quad \times\left[(\frac{r}{n-1} - 1)\frac{1}{n-2}\xi - (n-1)\right] + X\eta(Y)\left[(\frac{r}{n-1} - 1)\frac{1}{n-2}\eta(Z) - \frac{n-1}{n-2}\right] \\ &\quad + Y\eta(X)\left[(\frac{r}{n-1} - 1)\frac{1}{n-2}\eta(Z) - \frac{n-1}{n-2}\right]. \end{aligned} \tag{3.7}$$

For a conformally flat LP-Sasakian manifold, $R(X,Y)Z$ is given by the equation (3.7). Again in a LP-Sasakian manifold the following relation holds [9]

$$\begin{aligned} R(X,Y)\phi Z &= \phi(R(X,Y)Z) + 2\{\eta(X)Y - \eta(Y)X\}\eta(Z) + 2\{g(Y,Z)\eta(X) \\ &\quad -g(X,Z)\eta(Y)\}\xi - g(\phi X,Z)\phi Y + g(\phi Y,Z)\phi X - g(Y,Z)X \\ &\quad +g(X,Z)Y. \end{aligned} \tag{3.8}$$

Again, on using equations (1.15), (1.18) and (1.4) in (3.8) we calculate

$$g(\phi R(\phi X,\phi Y)Z,\phi W) = g(R(Z,W)\phi X,\phi Y).$$

Using (3.8) and then (1.7), (1.15) in the above equation we obtain

$$\begin{aligned} g(\phi R(\phi X,\phi Y)Z,\phi W) &= g(R(X,Y)Z,W) - g(W,X)\eta(Z)\eta(Y) + g(Z,X)\eta(W)\eta(Y) \\ &\quad +2\eta(Z)\eta(X)g(W,\phi Y) - 2\eta(W)\eta(X)g(Z,\phi Y) - g(\phi Z,X)g(\phi W,\phi Y) \\ &\quad +g(\phi W,X)g(\phi Z,\phi Y) - g(W,X)g(Z,\phi Y) + g(Z,X)g(W,\phi Y). \end{aligned} \tag{3.9}$$

Replacing X, Y by ϕX and ϕY respectively in (3.8) and taking inner product with ϕW we obtain on using (1.4) and (3.9) we get

$$\begin{aligned} g(R(\phi X,\phi Y)\phi Z,\phi W) &= g(R(X,Y)Z,W) - g(W,X)\eta(Z)\eta(Y) + g(Z,X)\eta(W)\eta(Y) \\ &\quad +3g(Y,\phi W)\eta(Z)\eta(X) - 3g(Z,\phi Y)\eta(W)\eta(X) + 2g(\phi W,X)g(Z,Y) \\ &\quad +2g(\phi W,X)\eta(Z)\eta(Y) - 2g(W,X)g(Z,\phi Y). \end{aligned} \tag{3.10}$$

Now we shall recall the definition of ϕ-section. A plane section in the tangent space $T_p(M)$ is called a ϕ-section if there exists a unit vector X in T_pM orthogonal to ξ such that $\{X, \phi X\}$ is an orthonormal basis of the plane section. Then the sectional curvature

$$K(X,\phi X) = g(R(X,\phi X)X,\phi X) \tag{3.11}$$

is called a ϕ-sectional curvature. A contact metric manifold $M(\phi,\xi,\eta,g)$ is said to be of constant ϕ-sectional curvature if at any point $P \in M$, the sectional curvature $K(X,\phi X)$ is independent of the choice of non-zero $X \in D_p$, where D denotes the contact distributions of the contact metric manifold defined by $\eta = 0$. The definition is valid for Lorentzian manifolds also [10].

We give the following theorem.

Theorem 3.1 *In a LP-Sasakian manifold $M(\phi,\xi,\eta,g)$ the relation $(Q\phi - \phi Q)X = 4n\phi X$ holds for any vector field X on M.*

Proof Let $\{X_i, \phi X_i, \xi\}$ $(i = 1, 2, \cdots, m)$ be a local $\phi-$basis at any point of the manifold. Now putting $Y = Z = X_i$ in (3.10) and taking summation over i, we obtain by virtue of $\eta(X_i) = 0$,

$$\Sigma\phi R(\phi X,\phi X_i)\phi X_i = \Sigma R(X,X_i)X_i + 2\phi X g(X_i,X_i). \tag{3.12}$$

Again setting $Y = Z = \phi X_i$ in (3.10) we have

$$\Sigma\phi R(\phi X,\phi^2 X_i)\phi^2 X_i = \Sigma R(X,\phi X_i)\phi X_i + 2\phi X g(X_i,X_i). \tag{3.13}$$

Adding (3.12) and (3.13) and using the definition of Ricci operator, we calculate

$$\phi(Q(\phi X) - R(\phi X, \xi)\xi) = QX - R(X, \xi)\xi + 4n\phi X. \tag{3.14}$$

We can write from (1.16)

$$R(\phi X, \xi)\xi = \phi X. \tag{3.15}$$

Using (3.13) and (3.14)

$$\phi(Q(\phi X)) = QX + 4n\phi X. \tag{3.16}$$

Operating ϕ on both sides and using (1.17)

$$Q(\phi X) - \phi(QX) = 4n\phi X. \tag{3.17}$$

By virtue of (3.17) theorem (3.1) is proved. □

For the next proof we consider the symbol W^i_{jk} where W^i_{jk} denotes the difference $\overline{\Gamma}^i_{jk} - \Gamma^i_{jk}$ of Christoffel symbols in an LP-Sasakian manifold [8]. In global notation we can write

$$W(Y, Z) = (1 - a)[\eta(Z)\phi Y + \eta(Y)\phi Z] + \frac{1}{2}(1 - \frac{1}{a})[(\nabla_Y \eta)Z + (\nabla_Z \eta)Y]\xi, \tag{3.18}$$

for all $Y, Z \in \chi(M)$. We state our next theorem.

Theorem 3.2 *Under a D-homothetic deformation, the operator $Q\phi - \phi Q$ of a LP-Sasakian manifold $M(\phi, \xi, \eta, g)$ is conformal.*

Proof If R and $\overline{R}$ denote the curvature tensors of the LP-Sasakian manifold $M(\phi, \xi, \eta, g)$ and $M(\overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$ respectively then we know from [8]

$$\begin{aligned}\overline{R}(X, Y)Z &= R(X, Y)Z + (\nabla_X W)(Z, Y) - (\nabla_Y W)(Z, X) \\ &\quad + W(W(Z, Y), X) - W(W(Z, X), Y).\end{aligned} \tag{3.19}$$

Using (1.13) in (3.18) we calculate

$$W(Y, Z) = (1 - a)[\eta(Z)\phi Y + \eta(Y)\phi Z] + (1 - \frac{1}{a})g(\phi Y, Z)\xi. \tag{3.20}$$

Taking covariant differentiation w.r.t. X and after using (1.8), (3.2), we obtain,

$$\begin{aligned}(\nabla_X W)(Y, Z) &= (1 - a)[g(\phi X, Z)\phi Y + g(X, Y)\eta(Z)\xi + 2\eta(Z)\eta(Y)X \\ &\quad + 4\eta(X)\eta(Y)\eta(Z)\xi + g(\phi X, Y)\phi Z + g(X, Z)\eta(Y)\xi] \\ &\quad + (1 - \frac{1}{a})g(\phi Y, Z)\phi X.\end{aligned} \tag{3.21}$$

Using (3.21) in (3.19) we obtain

$$\begin{aligned}\overline{R}(X, Y)Z &= R(X, Y)Z + (1 - a)\eta(Y)g(X, Z)\xi \\ &\quad + 2(1 - a)\eta(Z)\eta(Y)X + (1 - a)g(\phi X, Z)\phi Y + (1 - \frac{1}{a})g(\phi Z, Y)\phi X \\ &\quad - (1 - a)g(Y, Z)\eta(X)\xi \\ &\quad - 2(1 - a)\eta(X)\eta(Z)Y - (1 - a)g(\phi Y, Z)\phi X - (1 - \frac{1}{a})g(\phi Z, X)\phi Y\end{aligned}$$

$$+(1-a)\eta(Y)[(1-\frac{1}{a})g(\phi^2 Z,X)\xi]+(1-a)\eta(Z)[(1-a)\eta(X)\phi^2 Y$$
$$+(1-\frac{1}{a})g(\phi^2 Y,X)\xi]+(1-\frac{1}{a})g(\phi Z,X)[-(1-a)\phi X]$$
$$-(1-a)\eta(X)[(1-\frac{1}{a})g(\phi^2 Z,Y)\xi]-(1-a)\eta(Z)[(1-a)\eta(Y)\phi^2 X$$
$$+(1-\frac{1}{a})g(\phi^2 X,Y)\xi]-(1-\frac{1}{a})g(\phi Z,X)[-(1-a)\phi Y]. \tag{3.22}$$

From (3.22) we get

$$a\overline{S}(Y,Z)=S(Y,Z)+\frac{(1-a)^2}{a}. \tag{3.23}$$

Using the properties of Ricci operator

$$a\overline{Q}Y=QY+\frac{(1-a)^2}{a}.$$

Operating $\phi=\overline{\phi}$ on both sides from left hand side

$$a\overline{\phi}\ \overline{Q}Y=\phi QY+\frac{(1-a)^2}{a}.$$

Operating $\phi=\overline{\phi}$ on both sides from right hand side

$$a\overline{Q}\ \overline{\phi}Y=Q\phi Y+\frac{(1-a)^2}{a}.$$

Subtracting the above two equations we obtain

$$a(\overline{\phi}\ \overline{Q}-\overline{Q}\ \overline{\phi})=(\phi Q-Q\phi). \tag{3.24}$$

The equation (3.24) proves our theorem. □

We can also prove the following theorems as a consequence of D-homothetic deformation.

Theorem 3.3 *Under D-homothetic deformation, an $\eta-$Einstein LP-Sasakian manifold remains invariant.*

Proof In an η-Einstein LP-Sasakian manifold [9]

$$S(X,Y)=[\frac{r}{n-1}-1]g(X,Y)+[\frac{r}{n-1}-n]\eta(X)\eta(Y).$$

Under D-homothetic deformation we get

$$\overline{S}(X,Y)=[a(\frac{r}{n-1}-1)]g(X,Y)+[a(a-1)(\frac{r}{n-1}-1)+a^2(\frac{r}{n-1}-n)]\eta(X)\eta(Y).$$

Hence the result is proved. □

Theorem 3.4 *Under D-homothetic deformation, the ϕ-sectional curvature of a LP-Sasakian manifold is conformal.*

Proof Putting $Y=\phi X, Z=X$ in (3.12) and taking inner product with ϕX, we obtain on using (1.4) and the orthogonality property we get

$$ag(\overline{R}(X,\phi X)X,\phi X)=g(R(X,\phi X)X,\phi X)+(a-\frac{1}{a}) \tag{3.25}$$

$$a\overline{K}(X, \phi X) - K(X, \phi X) = (a - \frac{1}{a}). \qquad \square$$

Theorem 3.5 *There exists LP-Sasakian manifold with non-zero and non-constant ϕ-sectional curvature.*

Proof If the LP-Sasakian manifold satisfies $R(X, Y)\xi = 0$, then it can be proved easily that $K(X, \phi X) = 0$ and therefore from (3.25) we can conclude that $\overline{K}(X, \phi X) \neq 0$ for $a \neq 1$ where X is a unit vector field orthogonal to ξ. Hence the result is proved. $\square$

§4. An Example of a LP-Sasakian Manifold

In this section we shall prove the equality (3.25) by taking an example of LP-Sasakian manifold [1]. Let us consider a 5-dimensional manifold $\tilde{M} = \{(x, y, z, u, v) \in R^5 : ((x, y, z, u, v) \neq (0, 0, 0, 0, 0)\}$ where (x, y, z, u, v) are the standard coordinate in R^5. The vector fields

$$e_1 = -2\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}, \quad e_4 = -2\frac{\partial}{\partial u} + 2v\frac{\partial}{\partial z}, \quad e_5 = \frac{\partial}{\partial v}$$

are linearly independent at each point of M. Let g be the Lorentzian metric defined by

$$\begin{aligned} g(e_i, e_j) &= 1, \quad \text{for} \quad i = j \neq 3, \\ g(e_i, e_j) &= 0, \quad \text{for} \quad i \neq j, \\ g(e_3, e_3) &= -1. \end{aligned}$$

Here i and j runs from 1 to 5. Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$, for any vector field Z tangent to $\tilde{M}$. Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi e_1 = e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0, \quad \phi e_4 = e_5, \quad \phi e_5 = e_4.$$

Then using the linearity of ϕ and g we have

$$\eta(e_3) = -1, \quad \phi^2 Z = Z + \eta(Z)e_3,$$

for any vector fields Z, W tangent to $\tilde{M}$. Thus for $e_3 = \xi$, $\tilde{M}(\phi, \xi, \eta, g)$ forms a LP-Sasakian manifold.

Let ∇ be the Levi-Civita connection on $\tilde{M}$ with respect to the metric g. Then the followings can be obtained

$$[e_1, e_2] = -2e_3, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0.$$

On taking $e_3 = \xi$ and using Koszul's formula for the metric g, we calculate

$$\begin{aligned} &\nabla_{e_1} e_3 = e_2, \quad \nabla_{e_1} e_2 = -e_3, \quad \nabla_{e_1} e_1 = 0, \\ &\nabla_{e_2} e_3 = e_1, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_1 = e_3, \\ &\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = e_1, \quad \nabla_{e_3} e_1 = e_2. \end{aligned}$$

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

$$
\begin{aligned}
&R(e_1, e_2)e_2 = 3e_1, \quad R(e_1, e_2)e_1 = 3e_2, \quad R(e_2, e_3)e_3 = -e_2, \\
&R(e_1, e_3)e_2 = 0, \quad R(e_1, e_3)e_1 = -e_3, \quad R(e_2, e_3)e_2 = e_3, \\
&R(e_1, e_2)e_3 = 0.
\end{aligned}
$$

In equation (3.22) we put $X = e_1, Y = \phi e_1, Z = e_1$. Taking inner product with ϕe_1 we obtain

$$a\overline{K}(e_1, \phi e_1) - K(e_1, \phi e_1) = a - \frac{1}{a}.$$

Hence, by this example Theorem 3.4 is verified.

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Position Vectors of the Curves in Affine 3-Space According to Special Affine Frames

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Abstract: In literature, there are three affine frames commonly used for space curves, which are called equi-affine frame, Winternitz frame and intrinsic affine frame, respectively. In this study, we examined the position vectors of the space curves in affine 3-space for each of these three frames separately, in terms of lying in the planes $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ which are known as osculating, rectifying and normal planes, respectively and we obtained the position vectors and we gave some conclusions.

Key Words: Position vector, equi-affine frame, Winternitz frame, intrinsic affine frame.

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§1. Introduction

Affine differential geometry is the study of differential invariants with respect to the group of affine transformation. The group of affine motions special linear transformation namely the group of equi-affine or unimodular transformations consist of volume preserving ($\det(a_{jk}) = 1$) linear transformations together with translation such that

$$x_j^* = \sum_{k=1}^{3} a_{jk}x_k + c_j \qquad j = 1, 2, 3$$

This transformations group denoted by $ASL(3, IR) := SL(3, IR) \times IR^3$ and comprising diffeomorphisms of IR^3 that preserve some important invariants such curvatures that in curve theory as well. An equi-affine group is also called an Euclidean group [3].

Salkowski and Schells gave the equi-affine frame [4], Kreyszig and Pendl gave the characterization of spherical curves in both Euclidean and affine 3-spaces [3]. Su classified the curves in affine 3-space by using equi-affine frame [6]. Winternitz dwelled on the insufficiency of equi-affine frame for curves class of C^5 and defined the new frame known as Winternitz frame [5,1]. Davis obtained new affine frame by defining intrinsic affine binormal and in this study, we called that frame as intrinsic affine frame [2].

A set of points that corresponds to a vector of vector space constructed on a field is called an *affine space* associated with vector space. We denote A_3 as affine 3-space associated with IR^3. Let

$$\alpha : J \longrightarrow A_3$$

represent a curve in A_3, where $t \in J = (t_1, t_2) \subset IR$ is fixed and open interval. Regularity of a curve

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in A_3 is defined as$\begin{vmatrix} \dot{\alpha} & \ddot{\alpha} & \dddot{\alpha} \end{vmatrix} \neq 0$ on J, where $\dot{\alpha} = d\alpha/dt$, etc. Then, we may associate α with the invariant parameter

$$s = \sigma(t) = \int_{t_1}^{t} \begin{vmatrix} \dot{\alpha} & \ddot{\alpha} & \dddot{\alpha} \end{vmatrix}^{1/6} dt$$

which is called the *affine arc length* of $\alpha(s)$. The coordinates of a curve are given by three linearly independent solutions of the equations

$$\alpha^{(iv)}(s) + k(s)\,\alpha''(s) + \tau_\alpha(s)\,\alpha'(s) = 0 \tag{1}$$

under the condition

$$\begin{vmatrix} \alpha'(s) & \alpha''(s) & \alpha'''(s) \end{vmatrix} = 1 \tag{2}$$

where $k(s)$ and $\tau_\alpha(s)$ are differentiable functions of s.

§2. Position Vectors of the Curves in Affine 3-Space According to Equi-Affine Frame

Let $\alpha(s)$ be a regular curve with affine arc lengh parameter s. The vectors $\alpha'(s)$, $\alpha''(s)$ and $\alpha'''(s)$ are called tangent, affine normal and affine binormal vectors respectively, and the planes $sp\{\alpha'(s), \alpha''(s)\}$, $sp\{\alpha'(s), \alpha'''(s)\}$ and $sp\{\alpha''(s), \alpha'''(s)\}$ are called osculating, rectifying and normal planes of the curve $\alpha(s)$. Thus, the frame

$$\left\{ \begin{array}{lcl} T'(s) & = & N(s), \\ N'(s) & = & B(s), \\ B'(s) & = & -\tau_\alpha(s)T(s) - k(s)N(s) \end{array} \right. \tag{3}$$

is called equi-affine frame, where $k(s)$ and $\tau_\alpha(s)$ are called equi-affine curvature and equi-affine torsion, which are given as follow

$$k(s) = \begin{vmatrix} \alpha'(s) & \alpha'''(s) & \alpha^{(iv)}(s) \end{vmatrix} \tag{4}$$

$$\tau_\alpha(s) = -\begin{vmatrix} \alpha''(s) & \alpha'''(s) & \alpha^{(iv)}(s) \end{vmatrix}. \tag{5}$$

Let $f(s)$, $g(s)$ and $h(s)$ be differentiable functions then we can write

$$\alpha(s) = f(s)\,T(s) + g(s)\,N(s) + h(s)\,B(s) \tag{6}$$

and by differentiating equation (6) with respect to s and by using equations (3), we obtain

$$0 = \left\{ f'(s) - h(s)\,k_2(s) - 1 \right\} T(s) + \left\{ f(s) + g'(s) \right\} N(s) + \left\{ \begin{array}{c} h'(s) + g(s) \\ -h(s)\,k_1(s) \end{array} \right\} B(s)$$

Therefore, for $\alpha''(s) = N(s)$ and $B(s) = \alpha'''(s)$, we obtain the following theorem.

Theorem 2.1 *Let $\alpha(s)$ be a unit speed curve in A_3, with equi-affine curvature $k(s)$ and with equi-affine torsion $\tau_\alpha(s)$, then $\alpha(s)$ has the position vector in (6) according to equi-affine frame for some* *differentiable functions* $f(s)$, $g(s)$ *and* $h(s)$ *satisfy the equations*

$$\begin{cases} f'(s) - h(s)\,\tau_\alpha(s) & = & 1, \\ f(s) + g'(s) & = & 0, \\ h'(s) + g(s) - h(s)\,k(s) & = & 0. \end{cases}$$

Assume that the position vector of $\alpha(s)$ always lies in the plane $\mathrm{sp}\{N(s), B(s)\}$. Position vector of the curve $\alpha(s)$ satisfies the equation

$$\alpha(s) = g(s)\,N(s) + h(s)\,B(s) \tag{7}$$

for some differentiable functions $g(s)$ and $h(s)$. Differentiating equation (7) with respect to s, we obtain

$$0 = \{-h(s)\,\tau_\alpha(s) - 1\}\,T(s) + \{g'(s) - h(s)\,k(s)\}\,N + \{h'(s) + g(s)\}\,B(s)$$

It follows that

$$\begin{cases} h(s)\,\tau_\alpha(s) & = & -1, \\ g'(s) - h(s)\,k(s) & = & 0, \\ h'(s) + g(s) & = & 0 \end{cases} \tag{8}$$

and $h(s) = \frac{-1}{\tau_\alpha(s)}$, $g'(s) = -h''(s)$. Therefore, from the second equation we get

$$h''(s) + h(s)\,k(s) = 0 \tag{9}$$

and also

$$\left(\frac{1}{\tau_\alpha(s)}\right)'' + \frac{k(s)}{\tau_\alpha(s)} = 0, \tag{10}$$

and we find

$$\alpha(s) = \left(\frac{1}{\tau_\alpha(s)}\right)' N(s) - \frac{1}{\tau_\alpha(s)} B(s).$$

By considering $\alpha''(s) = N(s)$ and $\alpha'''(s) = B(s)$, we have the following theorem.

Theorem 2.2 *Let* $\alpha(s)$ *be a unit speed curve in* A_3 *, with nonzero equi-affine curvatures satisfying*

$$\left(\frac{1}{\tau_\alpha(s)}\right)'' + \frac{k(s)}{\tau_\alpha(s)} = 0,$$

then, $\alpha(s)$ *is a curve whose position vector according to equi-affine frame always lies in the sp*$\{N(s), B(s)\}$ *if and only if* $\alpha(s)$ *is the solution of the differential equation of*

$$\frac{1}{\tau_\alpha(s)}\alpha'''(s) - \left(\frac{1}{\tau_\alpha(s)}\right)' \alpha''(s) + \alpha(s) = 0.$$

In the case of $k(s) = 0$, from the first and the second equation of (8) $g(s) = c_0$, $h(s) = \frac{-1}{\tau_\alpha(s)}$ and from the third equation of (8), we get $\tau_\alpha(s) = \frac{1}{c_0 s - c_1}$. Thus, from (7), the position vector of $\alpha(s)$ satisfies the following differential equation

$$(c_0 s - c_1)\alpha'''(s) - c_0\alpha''(s) + \alpha(s) = 0.$$

In the case of $k(s)$ nonzero constant, from the second and the third equation of (8)

$$\begin{cases} g(s) & = & c_2\sqrt{k}\sin(\sqrt{k}s) - c_1\sqrt{k}\cos(\sqrt{k}s) \\ h(s) & = & c_1\sin(\sqrt{k}s) + c_2\cos(\sqrt{k}s) \end{cases}$$

and from the first equation of (8)

$$\tau_\alpha(s) = \frac{-1}{c_1\sin(\sqrt{k}s) + c_2\cos(\sqrt{k}s)}.$$

From (7), the position vector of $\alpha(s)$ satisfies the following differential equation

$$\left(c_2\sqrt{k}\sin(\sqrt{k}s) - c_1\sqrt{k}\cos(\sqrt{k}s)\right)\alpha''(s) + \left(c_1\sin(\sqrt{k}s) + c_2\cos(\sqrt{k}s)\right)\alpha'''(s) = \alpha(s)$$

It is clear that $\tau_\alpha(s)$ cannot be zero from the first equation of (8).

In the case of $\tau_\alpha(s)$ nonzero constant, from the first and the third equation of (8) $g(s) = 0$, $h(s) = \frac{-1}{\tau_\alpha}$ and from the second equation of (8), we obtain $k(s) = 0$. From (7), the position vector of $\alpha(s)$ satisfies the following differential equation

$$\alpha'''(s) + \tau_\alpha\alpha(s) = 0$$

$$\alpha(s) = c_1 e^{\frac{\sqrt[3]{-\tau_\alpha}bs}{2}} + c_2 e^{\frac{-\sqrt[3]{-\tau_\alpha}as}{2}} + c_3 e^{\sqrt[3]{-\tau_\alpha}s}.$$

In the case of $\tau_\alpha(s)$ and $k(s)$ nonzero constants, from the first and the third equation of (8) $g(s) = 0$, $h(s) = \frac{-1}{\tau_\alpha}$ and from the second equation of (8), we obtain $k(s) = 0$. By using (7), the position vector of $\alpha(s)$ satisfies the following differential equation

$$\alpha'''(s) + \tau_\alpha\alpha(s) = 0$$

$$\alpha(s) = c_1 e^{\frac{\sqrt[3]{-\tau_\alpha}bs}{2}} + c_2 e^{\frac{-\sqrt[3]{-\tau_\alpha}as}{2}} + c_3 e^{\sqrt[3]{-\tau_\alpha}s}$$

where a and b are scalars that can be complex in general. Hence, we know the following theorem.

Theorem 2.3 *Let $\alpha(s)$ be a unit speed curve in A_3, with the equi-affine curvature $k(s)$ and with the intrinsic affine torsion $\widetilde{\tau}_\alpha(s)$ whose position vector lies in $sp\{N(s), B(s)\}$ then the followings are true,*

(1) *If $k(s) = 0$ and $\tau_\alpha(s) = \frac{1}{c_0 s - c_1}$ then position vector of $\alpha(s)$ satisfies the equation*

$$(c_0 s - c_1)\alpha'''(s) - c_0\alpha''(s) + \alpha(s) = 0;$$

(2) *If $k(s) > 0$ constant and $\tau_\alpha(s) = \frac{-1}{\omega}$ then position vector of $\alpha(s)$ satisfies the equation*

$$\omega\alpha'''(s) - \omega'\alpha''(s) - \alpha(s) = 0,$$

where $\omega = c_1\sin(\sqrt{k}s) + c_2\cos(\sqrt{k}s)$;

(3) *There is no curve whose $\tau_\alpha(s) = 0$ in A_3;*

(4) *If $\tau_\alpha(s) < 0$ constant then $k(s) = 0$ and position vector of $\alpha(s)$ is*

$$\alpha(s) = c_1 e^{\rho_1 s} + c_2 e^{\rho_2 s} + c_3 e^{\rho_3 s}.$$

where $\rho_1 = \frac{\sqrt[3]{-\tau_\alpha}b}{2}$, $\rho_2 = \frac{-\sqrt[3]{-\tau_\alpha}a}{2}$ and $\rho_3 = \sqrt[3]{-\tau_\alpha}$. Here, a and b are scalars that can be complex in general.

We assume that the position vector of $\alpha(s)$ always lies in the plane sp$\{T(s), B(s)\}$. Position vector of the curve $\alpha(s)$ satisfies equation

$$\alpha(s) = f(s)\,T(s) + h(s)\,B(s) \tag{11}$$

for some differentiable functions $f(s)$ and $h(s)$. Differentiating equation (11) with respect to s, we obtain

$$0 = \left\{f'(s)\,T(s) - h(s)\,\tau_\alpha(s) - 1\right\}T + \left\{f(s) - h(s)\,k(s)\right\}N + h'(s)\,B(s).$$

It follows that

$$\begin{cases} f'(s) - h(s)\,\tau_\alpha(s) &= 1, \\ f(s) - h(s)\,k(s) &= 0, \\ h'(s) = 0 \end{cases}$$

and it is clear that$h(s) = c_0$ and then,

$$\begin{cases} f'(s) - c_0\tau_\alpha(s) &= 1, \\ f(s) - c_0 k(s) &= 0, \\ k'(s) - \tau_\alpha(s) &= \frac{1}{c_0}. \end{cases}$$

Therefore, we obtained

$$\alpha(s) = c_0 k_1(s) T(s) + c_0 B(s).$$

By considering $\alpha'(s) = T(s)$ and $\alpha'''(s) = B(s)$, we can give the following theorem.

Theorem 2.4 *Let $\alpha(s)$ be a unit speed curve in A_3 , with nonzero affine curvatures satisfying*

$$k'(s) - \tau_\alpha(s) = \frac{1}{c_0},$$

then, α is a curve whose position vector according to equi-affine frame always lies in the sp$\{T(s), B(s)\}$ if and only if α is the solution of the differential equation of

$$c_0 k(s)\alpha'(s) + c_0\alpha'''(s) - \alpha(s) = 0.$$

In the case of $k(s) = 0$, from the second and the third equation of (12)

$$h(s) = c_0, \quad f(s) = 0, \; \widetilde{\tau}_\alpha(s) \neq 0$$

and from the first equation of (12) we get

$$\tau_\alpha(s) = \frac{-1}{c_0}.$$

Thus, from (11), the position vector of $\alpha(s)$ satisfies the following differential equation

$$c_0\alpha'''(s) - \alpha(s) = 0$$

$$\alpha(s) = c_1 e^{\overline{\frac{-as}{2(c_0)^{1/3}}}} + c_2 e^{\overline{\frac{bs}{2(c_0)^{1/3}}}} + c_3 e^{\overline{\frac{s}{(c_0)^{1/3}}}}.$$

In the case of $k(s)$ nonzero constant, from the second and the third equation of (12) $h(s) = c_0$, $f(s) = c_0 k$ and from the first equation of (12) $\tau_\alpha(s) = \frac{-1}{c_0}$. From (11), the position vector of $\alpha(s)$

satisfies the following differential equation

$$c_0\alpha'''(s) + c_0 k\alpha'(s) - \alpha(s) = 0$$

$$\begin{aligned}\alpha(s) &= c_1 e^{\frac{\left\{-(c_0)^{1/3}2^{-1/3}a\lambda^2 + k\left\{-2\lambda(c_0)^{2/3} + c_0 k^3 2^{1/3} b\right\}\right\}s}{6(c_0)^{2/3}\lambda}} \\ &\quad + c_2 e^{\frac{-\left\{-(c_0)^{1/3}2^{-1/3}b\lambda^2 + k\left\{2\lambda(c_0)^{2/3} + c_0 k^3 2^{1/3} a\right\}\right\}s}{6(c_0)^{2/3}\lambda}} \\ &\quad + c_3 e^{\frac{\left\{(c_0)^{1/3}2^{-1/3}\lambda^2 + c_0 k^2 2^{1/3} - (c_0)^{2/3}k\lambda\right\}s}{3(c_0)^{2/3}\lambda}}\end{aligned}$$

where

$$\lambda = (-2k^3 c_0 + 3\sqrt{-12c_0 k^3 + 81} + 27)^{1/3}$$

and a, b are scalars that can be complex in general.

In the case of $\tau_\alpha(s) = 0$, from the second and the third equation of (12) $h(s) = c_0$, $f(s) = c_0 k(s)$ and from the first equation of (12), we obtain

$$k(s) = \frac{1}{c_0}s + c_1.$$

From (11), the position vector of $\alpha(s)$ satisfies the following differential equation

$$c_0\alpha'''(s) + c_0 k(s)\alpha'(s) - \alpha(s) = 0.$$

In the case of $\tau_\alpha(s)$ nonzero constant, from the second and the third equation of (12) $h(s) = c_0$, $f(s) = c_0 k(s)$ and from the first equation of (11), we obtain $k(s) = \frac{1+c_0\tau_\alpha}{c_0}s + c_1$. From (12), the position vector of $\alpha(s)$ satisfies the following differential equation

$$c_0\alpha'''(s)c_0 + k(s)\alpha'(s) - \alpha(s) = 0.$$

In the case of $\tau_\alpha(s)$ and $k(s)$ nonzero constants, from the second and the third equation of (12) $h(s) = c_0$, $f(s) = c_0 k$ and from the first equation of (12) we obtain $\tau_\alpha = \frac{1}{c_0}$. By using (11), the position vector of $\alpha(s)$ satisfies the following differential equation

$$c_0\alpha'''(s) + c_0 k\alpha'(s) - \alpha(s) = 0.$$

Hence, we obtain the following theorem.

Theorem 2.5 *Let $\alpha(s)$ be a unit speed curve in A_3 , with the equi-affine curvature $k(s)$ and with the intrinsic affine torsion $\widetilde{\tau}_\alpha(s)$ whose position vector lies in $sp\{N(s), B(s)\}$ then, the followings are true.*

(1) *If $k(s) = 0$ and $\tau_\alpha(s) = \frac{-1}{c_0}$ then position vector of $\alpha(s)$ satisfies the equation*

$$\alpha(s) = c_1 e^{\phi_1 s} + c_2 e^{\phi_2 s} + c_3 e^{\phi_3 s}$$

where $\phi_1 = \frac{-a}{2(c_0)^{1/3}}$, $\phi_2 = \frac{b}{2(c_0)^{1/3}}$ and $\phi_3 = \frac{1}{(c_0)^{1/3}}$;

(2) *If $\tau_\alpha(s)$ and $k(s)$ are nonzero constants then position vector of $\alpha(s)$ satisfies the equation*

$$\alpha(s) = c_1 e^{\varphi_1 s} + c_2 e^{\varphi_2 s} + c_3 e^{\varphi_3 s}$$

where

$$\begin{aligned}\varphi_1 &= \frac{\left\{-(c_0)^{1/3}2^{-1/3}a\lambda^2 + k\left\{-2\lambda(c_0)^{2/3} + c_0k^3 2^{1/3}b\right\}\right\}}{6(c_0)^{2/3}\lambda} \\ \varphi_2 &= \frac{-\left\{-(c_0)^{1/3}2^{-1/3}b\lambda^2 + k\left\{2\lambda(c_0)^{2/3} + c_0k^3 2^{1/3}a\right\}\right\}}{6(c_0)^{2/3}\lambda} \\ \varphi_3 &= \frac{\left\{(c_0)^{1/3}2^{-1/3}\lambda^2 + c_0k^2 2^{1/3} - (c_0)^{2/3}k\lambda\right\}}{3(c_0)^{2/3}\lambda},\end{aligned}$$

$$\lambda = (-2k^3c_0 + 3\sqrt{-12c_0k^3 + 81} + 27)^{1/3}$$

and $c_1, c_2, c_3 \in IR^3$ *such that* $\begin{vmatrix} c_1 & c_2 & c_3 \end{vmatrix} = 1$ *and a, b are scalars that can be complex in general;*

(3) *If* $\tau_\alpha(s) = 0$, *and* $k(s) = \frac{1}{c_0}s + c_1$ *or* $\tau_\alpha(s)$ *nonzero constant and* $k(s) = \frac{1+c_0\tau_\alpha}{c_0}s + c_1$ *then position vector of* $\alpha(s)$ *satisfies the equation*

$$c_0\alpha'''(s)c_0 + k(s)\alpha'(s) - \alpha(s) = 0.$$

Now, assume that the position vector of $\alpha(s)$ always lies in the plane sp$\{T(s), N(s)\}$. Position vector of the curve α satisfies equation

$$\alpha(s) = f(s)T(s) + g(s)N(s) \tag{13}$$

for some differentiable functions $f(s)$ and $g(s)$. Differentiating equation (13) with respect to s, we obtain

$$0 = \{f'(s) - 1\}T(s) + \{g'(s) + f(s)\}N(s) + g(s)B(s).$$

It follows that

$$\begin{cases} f'(s) &= 1, \\ g'(s) + f(s) &= 0, \\ g(s) &= 0. \end{cases} \tag{14}$$

There is no $f(s)$ and $g(s)$ satisfying equations (14). Thus, we get the following theorem.

Theorem 2.6 *There is no curve in* A_3 *whose position vector always lies in the sp*$\{T(s), N(s)\}$ *according to equi-affine frame.*

§3. Position Vectors of the Curves in Affine 3-Space According to Winternitz Frame

Let $\alpha(s)$ be regular C^5-curve with affine arclenght parameter s. A. Winternitz in [5] defined a new equi-affine frame by taking

$$T(s) = \alpha'(s), N(s) = \alpha''(s), \quad B(s) = \alpha'''(s) + \frac{k(s)}{4}\alpha'(s)$$

which are called tangent, affine normal, binormal vectors and

$$k_1(s) = \frac{-k(s)}{4}, \quad k_2(s) = \frac{k'(s)}{4} - \tau_\alpha(s)$$

which are called the first and the secod affine curvatures (also we called them the first and the second Winternitz affine curvatures). Here, $k(s)$ and $\tau_\alpha(s)$ are called equi-affine curvature and equi-affine torsion given in (4) and (5). Winternitz frame (also called equi-affine frame for C^5-curves) is defined with the equations

$$\begin{cases} T'(s) &= N(s) \\ N'(s) &= k_1(s)T(s)+B(s) \\ B'(s) &= k_2(s)T(s)+3k_1(s)N(s). \end{cases} \tag{15}$$

Let $f(s)$, $g(s)$ and $h(s)$ be differentiable functions, then we can write

$$\alpha(s)=f(s)\,T(s)+\ g(s)\,N(s)+h(s)\,B(s). \tag{16}$$

Differentiating equation (16) with respect to s and by using equations (15), we obtain

$$0=\left\{\begin{array}{c} f'(s)+\ g(s)\,k_1(s) \\ +h(s)\,k_2(s)-1 \end{array}\right\}T(s)+\left\{\begin{array}{c} g'(s)+f(s) \\ +3h(s)\,k_1(s) \end{array}\right\}N(s)+\left\{h'(s)+g(s)\right\}B(s)$$

Therefore, for $\alpha''(s)=N(s)$ and $B(s)=\alpha'''(s)+\frac{k(s)}{4}\alpha'(s)$, we get the following theorem.

Theorem 3.1 *Let $\alpha(s)$ be a unit speed curve in A_3, with Winternitz curvatures $k_1(s)$ and $k_2(s)$, then $\alpha(s)$ has the position vector in (17) according to Winternitz frame for some differentiable functions $f(s)$, $g(s)$ and $h(s)$ satisfies the equations*

$$\begin{cases} f'(s)+\ g(s)\,k_1(s)+h(s)\,k_2(s) &= 1 \\ g'(s)+f(s)+3h(s)\,k_1(s) &= 0 \\ h'(s)+g(s) &= 0. \end{cases}$$

Assume that the position vector of $\alpha(s)$ always lies in the plane $\mathrm{sp}\{N(s),B(s)\}$. Position vector of the curve $\alpha(s)$ satisfies the equation

$$\alpha(s)=\ g(s)\,N(s)+h(s)\,B(s) \tag{17}$$

for some differentiable functions $g(s)$ and $h(s)$. Differentiating equation (17) with respect to s, we obtain

$$0=\ \{g(s)\,k_1(s)+h(s)\,k_2(s)-1\}\,T(s)+\left\{g'(s)+3h(s)\,k_1(s)\right\}N(s)+\left\{h'(s)+\ g(s)\right\}B(s).$$

Thus, we have the following equations

$$\begin{cases} g(s)\,k_1(s)+h(s)\,k_2(s) &= 1 \\ g'(s)+3h(s)\,k_1(s) &= 0 \\ h'(s)+g(s) &= 0. \end{cases} \tag{18}$$

From the first and the third equation of (18)

$$h'(s)\,k_1(s)-h(s)\,k_2(s)+1=0$$

then solution is

$$h\left(s\right)=\varphi\left\{-\int\frac{ds}{\varphi k_1(s)}+c_0\right\},\tag{19}$$

where $\varphi=e^{\int\frac{k_2(s)}{k_1(s)}ds}$ and from the second equation for $g'\left(s\right)=-h''\left(s\right)$ we get

$$h''\left(s\right)-3h\left(s\right)k_1(s)=0.\tag{20}$$

Then by using (19), (20) it turns to

$$\varphi\left\{\int\frac{ds}{\varphi k_1(s)}-c_0\right\}\left\{3k_1(s)-\frac{\left(k_2(s)\right)^2}{\left(k_1(s)\right)^2}-\left\{\frac{k_2(s)}{k_1(s)}\right\}'\right\}-\frac{k_2(s)}{\left(k_1(s)\right)^2}+\frac{k_1'(s)}{\left(k_1(s)\right)^2}=0\tag{21}$$

and we find

$$\alpha\left(s\right)=\left\{\varphi'\int\frac{ds}{\varphi k_1(s)}-c_0\varphi'+\frac{\varphi}{\varphi k_1(s)}\right\}N(s)-\varphi\left\{\int\frac{ds}{\varphi k_1(s)}-c_0\right\}B(s).$$

By considering $\alpha''\left(s\right)=N(s)$ and $B(s)=\alpha'''\left(s\right)+\frac{k(s)}{4}\alpha'\left(s\right)$, we get the following theorem.

Theorem 3.2 *Let $\alpha\left(s\right)$ be a unit speed curve in A_3 , with nonzero Winternitz curvatures satisfing (21), then $\alpha(s)$ is a curve whose position vector according to Winternitz affine frame always lies in $sp\{N(s),B(s)\}$ if and only if $\alpha(s)$ is the solution of the differential equation of*

$$\left\{\begin{array}{r}\varphi\left\{\int\frac{ds}{\varphi k_1(s)}-c_0\right\}\alpha'''\left(s\right)-\left\{\varphi'\int\frac{ds}{\varphi k_1(s)}-c_0\varphi'+\frac{\varphi}{\varphi k_1(s)}\right\}\alpha''\left(s\right)\\-\varphi\left\{\int\frac{ds}{\varphi k_1(s)}-c_0\right\}k_1(s)\alpha'\left(s\right)+\alpha\left(s\right)\end{array}\right\}=0.$$

In the case of $k_1(s)=0$, from (18), we obtain $g\left(s\right)=c_0$, $h\left(s\right)=-c_0s+c_1$ and $k_2(s)=\frac{1}{-c_0s+c_1}$. From (17), position vector of $\alpha\left(s\right)$ satisfies

$$\left(c_0s-c_1\right)\alpha'''\left(s\right)-c_0\alpha''\left(s\right)+\alpha\left(s\right)=0.$$

In the case of $k_1(s)\neq 0$ constant, from the second and the third equation of (18), we get

$$h''\left(s\right)-3k_1h\left(s\right)=0$$

and the solution is

$$h\left(s\right)=c_1e^{s\sqrt{3k_1}}+c_2e^{-s\sqrt{3k_1}}.$$

Also, from the first equation, we get the second curvature is

$$k_2(s)=\frac{1+c_1\sqrt{3k_1}e^{s\sqrt{3k_1}}-c_2\sqrt{3k_1}e^{-s\sqrt{3k_1}}}{\left(c_1e^{s\sqrt{3k_1}}+c_2e^{-s\sqrt{3k_1}}\right)}$$

and so $g\left(s\right)$ is

$$g\left(s\right)=\sqrt{3k_1}\left\{c_2e^{-s\sqrt{3k_1}}-c_1e^{s\sqrt{3k_1}}\right\}.$$

From (17), position vector of $\alpha\left(s\right)$ satisfies

$$\left\{\begin{array}{c}\left(c_1e^{s\sqrt{3k_1}}+c_2e^{-s\sqrt{3k_1}}\right)\alpha'''\left(s\right)+\sqrt{3k_1}\left\{c_2e^{-s\sqrt{3k_1}}-c_1e^{s\sqrt{3k_1}}\right\}\alpha''\left(s\right)\\-k_1\left(c_1e^{s\sqrt{3k_1}}+c_2e^{-s\sqrt{3k_1}}\right)\alpha'\left(s\right)-\alpha\left(s\right)\end{array}\right\}=0.$$

In the case of $k_2(s)=0$, from the first and the third equation of (18), we obtain $g(s)=\frac{1}{k_1(s)}$, $h(s)=-\int\frac{ds}{k_1(s)}+c_0$ and from the second equation of (18), we obtain the relation of the curvatures

$$k_1''(s)k_1(s)-3k_1'(s)+3k_1(s)^3=0.$$

Thus, from (17), position vector of $\alpha(s)$ satisfies

$$\left(-\int\frac{ds}{k_1(s)}+c_0\right)\alpha'''(s)+\frac{1}{k_1(s)}\alpha''(s)-\left(-\int\frac{ds}{k_1(s)}+c_0\right)k_1(s)\alpha'(s)-\alpha(s)=0.$$

In the case of $k_2(s)$ nonzero constant, from the first and the third equation of (18), we obtain

$$h(s)=\left\{\frac{1}{k_2}+c_0e^{k_2\int\frac{ds}{k_1(s)}}\right\},\quad g(s)=-\frac{c_0k_2}{k_1(s)}e^{k_2\int\frac{ds}{k_1(s)}}$$

and from the second equation of (18), we obtain the relation between the curvatures as follows

$$\left\{\frac{3k_1(s)^3-(k_2)^2}{k_1(s)^2}\right\}c_0e^{k_2\int\frac{ds}{k_1(s)}}+\frac{3k_1(s)}{k_2}=0.$$

From (17), position vector of $\alpha(s)$ satisfies

$$\left\{\begin{array}{c}\left\{\frac{1}{k_2}+c_0e^{k_2\int\frac{ds}{k_1(s)}}\right\}\alpha'''(s)-\frac{c_0k_2}{k_1(s)}e^{k_2\int\frac{ds}{k_1(s)}}\alpha''(s)\\ -\left\{\frac{1}{k_2}+c_0e^{k_2\int\frac{ds}{k_1(s)}}\right\}k_1(s)\alpha'(s)-\alpha(s)\end{array}\right\}=0.$$

In the case of $k_1(s)$ and $k_2(s)$ nonzero constants, from the first and the third equation of (18), we get

$$h(s)=\frac{1}{k_2}+c_0e^{\frac{k_2}{k_1}s},\quad g(s)=-c_0\frac{k_2}{k_1}e^{\frac{k_2}{k_1}s}$$

and also from the second equation of (18), we get the relation between the curvatures as follows

$$\left\{3(k_1)^2-(k_2)^2\right\}c_0k_2e^{\frac{k_2}{k_1}s}+3(k_1)^3=0.$$

From (17), position vector of $\alpha(s)$ satisfies

$$\left(\frac{1}{k_2}+c_0e^{\frac{k_2}{k_1}s}\right)\alpha'''(s)\ -c_0\frac{k_2}{k_1}e^{\frac{k_2}{k_1}s}\alpha''(s)-\left(\frac{1}{k_2}+c_0e^{\frac{k_2}{k_1}s}\right)k_1\alpha'(s)-\alpha(s)=0.$$

Therefore, we get the following theorem.

Theorem 3.3 *Let $\alpha(s)$ be a unit speed curve in A_3 , with the Winternitz curvatures $k_1(s)$ and $k_2(s)$, whose position vector lies in $sp\{N(s),B(s)\}$ then, the followings are true.*

(1) *If $k_1(s)=0$ and $k_2(s)=\frac{1}{-c_0s+c_1}$ then position vector of $\alpha(s)$ satisfies the equation*

$$(c_0s-c_1)\alpha'''(s)-c_0\alpha''(s)+\alpha(s)=0;$$

(2) *If $k_1(s)>0$ is constant and $k_2(s)=\frac{1+\widetilde{\varphi}'}{\widetilde{\varphi}}$ then position vector of $\alpha(s)$ satisfies the equation*

$$\widetilde{\varphi}v\alpha'''(s)-\widetilde{\varphi}'\alpha''(s)-k_1\widetilde{\varphi}\alpha'(s)-\alpha(s)=0$$

where $\widetilde{\varphi}=c_1e^{s\sqrt{3k_1}}+c_2e^{-s\sqrt{3k_1}}$;

(3) *If $k_2(s)=0$ and $k_1(s)$ satisfy $k_1''(s)k_1(s)-3k_1'(s)+3k_1(s)^3=0$ then position vector of $\alpha(s)$*

satisfies the equation

$$\psi\alpha'''(s)+\frac{1}{k_1(s)}\alpha''(s)-\psi k_1(s)\alpha'(s)-\alpha(s)=0$$

where $\psi=-\int\frac{ds}{k_1(s)}+c_0$;

(4) *If* $k_2(s)$ *is nonzero constant,* $k_1(s)$ *and* k_2 *satisfy*

$$\left\{\frac{3k_1(s)^3-(k_2)^2}{k_1(s)^2}\right\}c_0e^{k_2\int\frac{ds}{k_1(s)}}+\frac{3k_1(s)}{k_2}=0$$

then, the position vector of $\alpha(s)$ *satisfies the equation*

$$\left\{\frac{1}{k_2}+c_0\phi\right\}\alpha'''(s)-\frac{c_0\phi k_2}{k_1(s)}\alpha''(s)-\left\{\frac{1}{k_2}+c_0\phi\right\}k_1(s)\alpha'(s)-\alpha(s)=0$$

where $\phi=e^{k_2\int\frac{ds}{k_1(s)}}$;

(5) *If* $k_1(s)$, $k_2(s)$ *nonzero constants and satisfy the equation*

$$\left\{3(k_1)^2-(k_2)^2\right\}c_0k_2e^{\frac{k_2}{k_1}s}+3(k_1)^3=0$$

then, the position vector of $\alpha(s)$ *satisfies the equation*

$$\left(\frac{1}{k_2}+c_0\upsilon\right)\alpha'''(s)\ -c_0\frac{k_2}{k_1}\upsilon\alpha''(s)-\left(\frac{1}{k_2}+c_0\upsilon\right)k_1\alpha'(s)-\alpha(s)=0$$

where $\upsilon=e^{\frac{k_2}{k_1}s}$.

We assume that the position vector of $\alpha(s)$ always lies in the plane $\mathrm{sp}\{T(s),B(s)\}$. Position vector of the curve $\alpha(s)$ satisfies equation

$$\alpha(s)=f(s)T(s)+h(s)B(s) \tag{22}$$

for some differentiable functions $f(s)$ and $h(s)$. Differentiating equation (22) with respect to s, we obtain

$$0=\left\{f'(s)+h(s)k_2(s)-1\right\}T(s)+\{f(s)+3h(s)k_1(s)\}N(s)+h'(s)B(s),$$

it follows that

$$\begin{cases} f'(s)+h(s)k_2(s)=1\\ f(s)+3h(s)k_1(s)=0\\ h'(s)=0\end{cases} \tag{23}$$

for $h(s)=c_0$. From the first and second equations of (23), we get

$$k_2(s)-3k_1'(s)=\frac{1}{c_0}$$

and

$$f(s)=-3c_0k_1(s).$$

Therefore, we obtained

$$\alpha(s)=-3c_0k_1(s)T(s)+c_0B(s).$$

By considering $\alpha'(s)=T(s)$ and $B(s)=\alpha'''(s)+\frac{k(s)}{4}\alpha'(s)$, we get the following theorem.

Theorem 3.4 *Let $\alpha(s)$ be a unit speed curve in A_3 with nonzero Winternitz curvatures satisfing $k_2(s) - 3k_1'(s) = \frac{1}{c_0}$, then, $\alpha(s)$ is a curve whose position vector according to Winternitz affine frame always lies in $sp\{T(s), B(s)\}$ if and only if $\alpha(s)$ is the solution of the differential equation of*

$$c_0\alpha'''(s) - 4c_0k_1(s)\alpha'(s) - \alpha(s) = 0$$

In the case of $k_1(s) = 0$, from (23), we find $h(s) = c_0$, $f(s) = 0$ and $k_2(s) \neq 0$. From (22), we get

$$c_0\alpha'''(s) - \alpha(s) = 0$$

and the solution is

$$\alpha(s) = c_1 e^{\frac{-as}{2(c_0)^{1/3}}} + c_2 e^{\frac{bs}{2(c_0)^{1/3}}} + c_3 e^{\frac{s}{(c_0)^{1/3}}}.$$

In the case of $k_1(s)$ nonzero constant, from the second and the third equation of (23), we obtained $h(s) = c_0$, $f(s) = -3c_0k_1$ and also $k_2(s) = \frac{1}{c_0}$. From (22), we get $\alpha(s)$ satisfies the equation

$$c_0\alpha'''(s) - 4c_0k_1\alpha'(s) - \alpha(s) = 0.$$

In the case of $k_2(s) = 0$ constant, from the first and the third equation of (23) $f(s) = s + c_1$, $h(s) = c_0$ and from the second equation of (23), we obtain $k_1(s) = \frac{s+c_1}{-3c_0}$. From (22), we get that $\alpha(s)$ satisfies the equation

$$3c_0\alpha'''(s) + 4(s + c_1)\alpha'(s) - 3\alpha(s) = 0.$$

In the case of $k_2(s)$ nonzero constant, from the first and the third equation of (23) $h(s) = c_0$, $f(s) = (1 - c_0k_2)s + c_1$ and from the second equation of (23), we obtain $k_1(s) = \frac{(c_0k_2-1)s-c_1}{3c_0}$. From (22), we get that $\alpha(s)$ satisfies the equation

$$3c_0\alpha'''(s) + 4\left((1 - c_0k_2)s + c_1\right)\alpha'(s) - 3\alpha(s) = 0.$$

In the case of $k_1(s)$ and $k_2(s)$ nonzero constants, from the first and the third equation of (23) $h(s) = c_0$, $f(s) = -3c_0k_1$ and also from the second equation of (23), we obtain $k_2 = \frac{-1}{c_0}$. From (22), we get

$$c_0\alpha'''(s) - 4c_0k_1\alpha'(s) - \alpha(s) = 0$$

and the solution is

$$\begin{aligned} \alpha(s) \quad = \quad & c_1 e^{\frac{\left\{4k_1(c_0)^{2/3}b12^{1/3} - a\varphi^2\right\}s}{12\varphi}} + c_2 e^{\frac{-\left\{4k_1(c_0)^{2/3}12^{1/3}a - \varphi^2 b\right\}s}{12\varphi}} \\ & + c_3 e^{\frac{12^{1/3}\left\{4k_1(c_0)^{2/3}12^{1/3} + \varphi^2\right\}s}{6\varphi(c_0)^{1/3}}}, \end{aligned}$$

where $\varphi = \left\{9 + \sqrt{-768(c_0)^2(k_1)^3 + 81}\right\}^{1/3}$. Hence, we can give the following theorem.

Theorem 3.5 *Let $\alpha(s)$ be a unit speed curve in A_3 with the Winternitz curvatures $k_1(s)$ and $k_2(s)$, whose position vector lies in $sp\{N(s), B(s)\}$, then, the followings are true.*

(1) *If $k_1(s) = 0$ then position vector of $\alpha(s)$ is*

$$\alpha(s) = c_1 e^{r_1 s} + c_2 e^{r_2 s} + c_3 e^{r_3 s}$$

for some $k_2(s)$, where $r_1 = \frac{-a}{2(c_0)^{1/3}}$, $r_2 = \frac{-b}{2(c_0)^{1/3}}$ and $r_3 = \frac{1}{(c_0)^{1/3}}$;

(2) *If $k_1(s)$ and $k_2(s)$ are nonzero constants then, position vector of $\alpha(s)$ is*

$$\alpha(s) = c_1 e^{\psi_1 s} + c_2 e^{\psi_2 s} + c_3 e^{\psi_3 s},$$

where

$$\begin{aligned}
\psi_1 &= \frac{\left\{4k_1(c_0)^{2/3} b 12^{1/3} - a\varphi^2\right\}}{12\varphi} \\
\psi_2 &= \frac{-\left\{4k_1(c_0)^{2/3} 12^{1/3} a - \varphi^2 b\right\}}{12\varphi} \\
\psi_3 &= \frac{12^{1/3}\left\{4k_1(c_0)^{2/3} 12^{1/3} + \varphi^2\right\}}{6\varphi(c_0)^{1/3}}
\end{aligned}$$

$$\varphi = \left\{9 + \sqrt{-768(c_0)^2(k_1)^3 + 81}\right\}^{1/3}$$

and a, b are scalars that can be complex in general;

(3) *If $k_2(s) = 0$ and $k_1(s) = \frac{s+c_1}{-3c_0}$ then, position vector of $\alpha(s)$ satisfies the equation*

$$3c_0\alpha'''(s) + 4(s+c_1)\alpha'(s) - 3\alpha(s) = 0;$$

(4) *If $k_2(s)$ is nonzero constant and $k_1(s) = \frac{(c_0k_2-1)s-c_1}{3c_0}$ then, position vector of $\alpha(s)$ satisfies the equation*

$$3c_0\alpha'''(s) + 4\left((1-c_0k_2)s + c_1\right)\alpha'(s) - 3\alpha(s) = 0.$$

Now, assume that the position vector of $\alpha(s)$ always lies in the plane sp$\{T(s), N(s)\}$. Position vector of the curve $\alpha(s)$ satisfies equations

$$\alpha(s) = f(s)\,T(s) + g(s)\,N(s) \tag{24}$$

and

$$0 = \left\{f'(s) + g(s)\,k_1(s) - 1\right\}T(s) + \left\{g'(s) + f(s)\right\}N(s) + g(s)\,B(s)$$

for some differentiable functions $f(s)$ and $g(s)$. Differentiating equation (24) with respect to s, we obtain

$$\left\{\begin{array}{lcl}
f'(s) + g(s)\,k_1(s) & = & 1 \\
g'(s) + f(s) & = & 0 \\
g(s) & = & 0.
\end{array}\right. \tag{25}$$

There is no $f(s)$ and $g(s)$ satisfying equations (25). Thus, we get the following theorem.

Theorem 3.6 *There is no curve in A_3 whose position vector always lies in sp$\{T(s), N(s)\}$ according to Winternitz affine frame.*

§4. Position Vectors of the Curves in Affine 3-Space According to Intrinsic Equi-Affine Frame

In [2], D. Davis obtained a new affine frame by taking $T(s) := \alpha'(s)$, $N(s) := \alpha''(s)$, $B(s) := k(s)\alpha'(s) + \alpha'''(s)$ (which is called intrinsic affine binormal) and $\widetilde{\tau}_\alpha(s) := k(s) - \tau'_\alpha(s)$ (which is called intrinsic affine torsion) with the equations

$$\begin{cases} T'(s) & = & N(s) \\ N'(s) & = & -k(s)T(s) + B(s) \\ B'(s) & = & -\widetilde{\tau}_\alpha(s)T(s). \end{cases} \tag{26}$$

We called $\{T(s), N(s), B(s)\}$ is intrinsic affine frame. Here, $k(s)$ and $\tau_\alpha(s)$ are called equi-affine curvature and equi-affine torsion given in (4) and (5).

Let $f(s)$, $g(s)$ and $h(s)$ be differentiable functions then, we can write

$$\alpha(s) = f(s)\,T(s) + \; g(s)\,N(s) + h(s)\,B(s). \tag{27}$$

Differentiating equation (27) with respect to s and by using equations (26), we obtain

$$0 = \left\{f'(s) - h(s)\,\widetilde{\tau}_\alpha(s) - 1\right\}T(s) + \left\{\; f(s) + g'(s)\right\}N(s) + \left\{\begin{array}{c} h'(s) + \; g(s) \\ -h(s)\,k(s) \end{array}\right\}B(s).$$

For $\alpha''(s) = N(s)$ and $B(s) = k(s)\alpha'(s) + \alpha'''(s)$, we can give the following theorem.

Theorem 4.1 *Let $\alpha(s)$ be a unit speed curve in A_3 with equi-affine curvature $k(s)$ and with intrinsic torsion $\widetilde{\tau}_\alpha(s)$, then, $\alpha(s)$ has the position vector in* (27) *according to intrinsic equi-affine frame for some differentiable functions $f(s)$, $g(s)$ and $h(s)$ satisfy the equations*

$$\begin{cases} f'(s) - h(s)\,\widetilde{\tau}_\alpha(s) & = & 1, \\ f(s) + g'(s) & = & 0, \\ h'(s) + \; g(s) - h(s)\,k(s) & = & 0. \end{cases}$$

Assume that the position vector of $\alpha(s)$ always lies in the plane $\mathrm{sp}\{N(s), B(s)\}$ then, position vector of the curve $\alpha(s)$ satisfies the equation

$$\alpha(s) = \; g(s)\,N(s) + h(s)\,B(s) \tag{28}$$

for some differentiable functions $g(s)$ and $h(s)$. Differentiating equation (28) with respect to s, we obtain

$$0 = \; \{-h(s)\,\widetilde{\tau}_\alpha(s) - g(s)\,k(s) - 1\}\,T(s) + g'(s)\,N(s) + \left\{h'(s) + g(s)\right\}B(s).$$

It follows that

$$\begin{cases} h(s)\,\widetilde{\tau}_\alpha(s) + g(s)\,k(s) & = & -1, \\ g'(s) & = & 0, \\ h'(s) + g(s) & = & 0. \end{cases} \tag{29}$$

and we get $g(s) = c_0$ and $h(s) = -c_0 s + c_1$. From the second, the third and the first equation of (29), we obtain

$$(-c_0 s + c_1)\,\widetilde{\tau}_\alpha(s) + c_0 k(s) = -1.$$

In this case, we can write

$$\alpha(s) = \; c_0 N(s) + (-c_0 s + c_1)\,B(s).$$

By considering $\alpha''(s) = N(s)$ and $B(s) = k(s)\alpha'(s) + \alpha'''(s)$ we can give the following theorem.

Theorem 4.2 *Let $\alpha(s)$ be a unit speed curve in A_3 with nonzero equi-affine curvature $k(s)$ and with intrinsic torsion $\widetilde{\tau}_\alpha(s)$ satisfying*

$$(c_0 s - c_1)\,\widetilde{\tau}_\alpha(s) - c_0 k(s) = 1,$$

then, $\alpha(s)$ is a curve whose position vector according to intrinsic equi-affine frame always lies in $sp\{N(s), B(s)\}$ if and only if $\alpha(s)$ is the solution of the equation

$$(-c_0 s + c_1)\,\alpha'''(s) + c_0\alpha''(s) + (-c_0 s + c_1)\,k(s)\alpha'(s) - \alpha(s) = 0.$$

In the case of $k(s) = 0$, from the first and the second equation of (29), we obtained $g(s) = c_0$, $h(s) = \frac{-1}{\widetilde{\tau}_\alpha(s)}$ and $\widetilde{\tau}_\alpha(s) \neq 0$. From the third equation of (29), we get $\widetilde{\tau}_\alpha(s) = \frac{1}{c_0 s - c_1}$. Thus, from (28), the position vector of $\alpha(s)$ satisfies the following differential equation

$$\alpha'''(s) - c_0\widetilde{\tau}_\alpha(s)\alpha''(s) + \widetilde{\tau}_\alpha(s)\alpha(s) = 0.$$

In the case of $k(s)$ nonzero constant, from the second and the third equation of (29), we obtained $g(s) = c_0$, $h(s) = -c_0 s + c_1$. From the first equation of (29), we get $\widetilde{\tau}_\alpha(s) = \frac{1+c_0 k}{c_0 s - c_1}$. From (28), the position vector of $\alpha(s)$ satisfies the following differential equation

$$(-c_0 s + c_1)\,\alpha'''(s) + c_0\alpha''(s) + (-c_0 s + c_1)\,k\alpha'(s) - \alpha(s) = 0.$$

In the case of $\widetilde{\tau}_\alpha(s) = 0$, from the second and the third equation of (29), we obtained $g(s) = c_0$, $h(s) = -c_0 s + c_1$. From the first equation of (29), we obtained $k(s) = \frac{-1}{c_0}$. From (28), the position vector of $\alpha(s)$ satisfies the following differential equation

$$(-c_0 s + c_1)\,\alpha'''(s) + c_0\alpha''(s) + (-c_0 s + c_1)\,k(s)\alpha'(s) - \alpha(s) = 0.$$

In the case of $\widetilde{\tau}_\alpha(s)$ nonzero constant, from the second and the third equation of (29) $g(s) = c_0$, $h(s) = -c_0 s + c_1$ and from the first equation of (29), we obtain $k(s) = \frac{(c_0 s - c_1)\widetilde{\tau}_\alpha - 1}{c_0}$. From (28), the position vector of $\alpha(s)$ satisfies the following differential equation

$$(-c_0 s + c_1)\,\alpha'''(s) + c_0\alpha''(s) + (-c_0 s + c_1)\,k(s)\alpha'(s) - \alpha(s) = 0.$$

In the case of $\widetilde{\tau}_\alpha(s)$ and $k(s)$ nonzero constants, from the second and the third equation of (29), we obtained $g(s) = c_0$, $h(s) = -c_0 s + c_1$ and from the first equation of (29), we obtained $\widetilde{\tau}_\alpha = \frac{1+c_0 k}{c_0 s - c_1}$. By using (28), the position vector of $\alpha(s)$ satisfies the following differential equation,

$$(-c_0 s + c_1)\,\alpha'''(s) + c_0\alpha''(s) + (-c_0 s + c_1)\,k\alpha'(s) - \alpha(s) = 0.$$

Hence, we get the following theorem.

Theorem 4.3 *Let $\alpha(s)$ be a unit speed curve in A_3 with the equi-affine curvature $k(s)$ and with the intrinsic affine torsion $\widetilde{\tau}_\alpha(s)$ whose position vector lies in $sp\{N(s), B(s)\}$ then, the followings are true.*

(1) *If $k(s) = 0$ then position vector of $\alpha(s)$ satisfies the equation*

$$\alpha'''(s) - c_0\widetilde{\tau}_\alpha(s)\alpha''(s) + \widetilde{\tau}_\alpha(s)\alpha(s) = 0$$

for some $\widetilde{\tau}_\alpha(s)$;

(2) *If* $\widetilde{\tau}_\alpha(s) = 0$ *and* $k(s) = \frac{-1}{c_0}$ *or* $\widetilde{\tau}_\alpha(s)$ *is nonzero constant and*

$$k(s) = \frac{(c_0 s - c_1)\widetilde{\tau}_\alpha - 1}{c_0}$$

then, position vector of $\alpha(s)$ *satisfies the equation*

$$(-c_0 s + c_1)\,\alpha'''(s) + c_0\alpha''(s) + (-c_0 s + c_1)\,k(s)\alpha'(s) - \alpha(s) = 0;$$

(3) *If* $k(s)$ *is nonzero constant and* $\widetilde{\tau}_\alpha(s) = \frac{1+c_0 k}{c_0 s - c_1}$ *or* $\widetilde{\tau}_\alpha(s)$ *and* $k(s)$ *are nonzero constants then, position vector of* $\alpha(s)$ *satisfies the equation*

$$(-c_0 s + c_1)\,\alpha'''(s) + c_0\alpha''(s) + (-c_0 s + c_1)\,k\alpha'(s) - \alpha(s) = 0.$$

We assume that the position vector of α always lies in the plane $\mathrm{sp}\{T(s), B(s)\}$. Position vector of the curve α satisfies equation

$$\alpha(s) = f(s)\,T(s) + h(s)\,B(s) \tag{30}$$

for some differentiable functions $f(s)$ and $h(s)$. Differentiating equation (30), with respect to s, we obtain

$$0 = \left\{f'(s) - h(s)\,\tau_\alpha(s) - 1\right\} T(s) + f(s)\,N(s) + h'(s)\,B(s).$$

It follows that

$$\begin{cases} f'(s) - h(s)\,\tau_\alpha(s) & = & 1, \\ f(s) & = & 0, \\ h'(s) & = & 0 \end{cases} \tag{31}$$

and we get $h(s) = c_0$ and $\tau_\alpha(s) = \frac{-1}{c_0}$. Thus, we can write

$$\alpha(s) = c_0 B(s).$$

If $\tau_\alpha(s) = 0$ then, there is no function $f(s)$ that satisfies the first and the second equation of (31). By considering $\alpha'(s) = T(s)$ and $B(s) = k(s)\alpha' + \alpha'''$, we can give the following theorem.

Theorem 4.4 *Let* $\alpha(s)$ *be a unit speed curve in* A_3 *with nonzero intrinsic affine torsion, then,* $\alpha(s)$ *is a curve whose position vector according to intrinsic equi-affine frame always lies in* $sp\{T(s), B(s)\}$ *if and only if* $\alpha(s)$ *is the solution of the equation*

$$c_0\alpha'''(s) + c_0 k(s)\alpha'(s) - \alpha(s) = 0$$

for some $k(s)$.

Now, assume that the position vector of $\alpha(s)$ always lies in the plane $\mathrm{sp}\{T(s), N(s)\}$. The position vector of the curve $\alpha(s)$ satisfies equation

$$\alpha(s) = f(s)\,T(s) + \ g(s)\,N(s) \tag{32}$$

for some differentiable functions $f(s)$ and $g(s)$. Differentiating equation (32) with respect to s, we obtain

$$0 = \left\{ f'(s)\, T(s) - g(s)\, \kappa_\alpha(s) - 1 \right\} T(s) + \left\{ g'(s) + f(s) \right\} N(s) + g(s)\, B(s).$$

It follows that

$$\begin{cases} f'(s)\, T(s) - g(s)\, \kappa_\alpha(s) & = & 1, \\ g'(s) + f(s) & = & 0, \\ g(s) & = & 0. \end{cases} \tag{33}$$

There is no $f(s)$ and $g(s)$ satisfying equations (33). Thus, we get the following theorem.

Theorem 4.5 *There is no curve in A_3 whose position vector always lies in the sp$\{T(s), N(s)\}$ according to intrinsic equi-affine frame.*

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A Generalization of Some Integral Inequalities for Multiplicatively P-Functions

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Abstract: In this paper, using identity for differentiable functions we can derive a general inequality containing all of the midpoint, trapezoid and Simpson inequalities for functions whose derivatives in absolute value at certain power are multiplicatively P-functions. Some applications to special means of real numbers are also given.

Key Words: Convex function, multiplicatively P-functions, Simpson's inequality, Hermite-Hadamard's inequality, midpoint inequality, trapezoid ineqaulity.

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§1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then the function f is said to be concave on interval $I \neq \emptyset$.

This definition is well known in the literature. It is well known that theory of convex sets and convex functions play an important role in mathematics and the other pure and applied sciences. In recent years, the concept of convexity has been extended and generalized in various directions using novel and innovative techniques.

Theorem 1.1 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions.

The classical Hermite-Hadamard integral inequality provides estimates of the mean value of a continuous convex or concave function. See [2-4, 7, 9], for the results of the generalization, improvement and extension of the famous integral inequality (1.1).

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The following inequality is well known in the literature as Simpson's inequality:

Let $f:[a,b]\rightarrow\mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_{\infty}=\sup\limits_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then the following inequality holds:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int\limits_{a}^{b}f(x)dx\right|\leq\frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^{2}.$$

In recent years many researchers have studied error estimations for Simpson's inequality; for refinements, counterparts, generalizations and new Simpson's type inequalities, see [1,10-12].

In this paper, in order to provide a unified approach to midpoint inequality, trapezoid inequality and Simpson's inequality for functions whose derivatives in absolute value at certain power are multiplicatively P-functions, we derive a general integral identity for differentiable functions. Finally some applications for special means of real numbers are provided.

Definition 1.2 *Let $I\neq\emptyset$ be an interval in $\mathbb{R}$. The function $f:I\rightarrow[0,\infty)$ is said to be multiplicatively P-function (or* log*-P-function), if the inequality*

$$f\left(tx+(1-t)y\right)\leq f(x)f(y)$$

holds for all $x,y\in I$ and $t\in[0,1]$.

In [8], some inequalities of Hermite-Hadamard type for differentiable multiplicatively P-functions were presented as follows.

Theorem 1.3 *Let the function $f:I\rightarrow[1,\infty)$,be a multiplicatively P-function and $a,b\in I$ with $a<b$. If $f\in L[a,b]$, then the following inequalities hold:*

(i) $f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_{a}^{b}f(x)f\left(a+b-x\right)dx\leq\left[f(a)f(b)\right]^{2}$;

(ii) $f\left(\frac{a+b}{2}\right)\leq f(a)f(b)\frac{1}{b-a}\int_{a}^{b}f(x)dx\leq\left[f(a)f(b)\right]^{2}$.

In [5], İşcan obtained inequalities for differentiable convex functions using following lemma.

Lemma 1.4 *Let $f:I\subset\mathbb{R}\rightarrow\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$ and $\alpha,\lambda\in[0,1]$. Then the following equality holds:*

$$\lambda\left(\alpha f(a)+(1-\alpha)f(b)\right)+(1-\lambda)f(\alpha a+(1-\alpha)b)-\frac{1}{b-a}\int\limits_{a}^{b}f(x)dx \tag{1.2}$$
$$=(b-a)\left[\int\limits_{0}^{1-\alpha}(t-\alpha\lambda)f'\left(tb+(1-t)a\right)dt+\int\limits_{1-\alpha}^{1}\left(t-1+\lambda(1-\alpha)\right)f'\left(tb+(1-t)a\right)dt\right].$$

§2. Main Results

In this section, in order to prove our main theorems, we shall use the identity (1.4).

Theorem 2.1 *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$ and $\alpha, \lambda \in [0,1]$. If $|f'|^q$ is multiplicatively P-function on $[a,b]$, $q \geq 1$, then the following inequality holds:*

$$\left|\lambda\left(\alpha f(a) + (1-\alpha) f(b)\right) + (1-\lambda) f(\alpha a + (1-\alpha) b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \tag{2.1}$$

$$\leq \begin{cases} (b-a)\left|f'(a)\right|\left|f'(b)\right|\left[\gamma_2 + \upsilon_2\right], & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ (b-a)\left|f'(a)\right|\left|f'(b)\right|\left[\gamma_2 + \upsilon_1\right] & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ (b-a)\left|f'(a)\right|\left|f'(b)\right|\left[\gamma_1 + \upsilon_2\right] & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases}$$

where

$$\gamma_1 = (1-\alpha)\left[\alpha\lambda - \frac{(1-\alpha)}{2}\right], \ \gamma_2 = (\alpha\lambda)^2 - \gamma_1 \ , \tag{2.2}$$

$$\upsilon_1 = \frac{1-(1-\alpha)^2}{2} - \alpha\left[1-\lambda(1-\alpha)\right], \tag{2.3}$$

$$\upsilon_2 = \frac{1+(1-\alpha)^2}{2} - (\lambda+1)(1-\alpha)\left[1-\lambda(1-\alpha)\right],$$

Proof Suppose that $q \geq 1$. From Lemma 1.4, the well known power mean inequality and property of multiplicatively P-function of $|f'|^q$ on $[a,b]$, that is

$$\left|f'\left(tb+(1-t)a\right)\right|^q \leq \left|f'(b)\right|^q \left|f'(a)\right|^q, \qquad t \in [0,1],$$

we have

$$\begin{aligned} &\left|\lambda\left(\alpha f(a) + (1-\alpha) f(b)\right) + (1-\lambda) f(\alpha a + (1-\alpha) b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\leq (b-a)\left[\int_0^{1-\alpha} |t-\alpha\lambda| \left|f'\left(tb+(1-t)a\right)\right| dt + \int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)| \left|f'\left(tb+(1-t)a\right)\right| dt\right] \\ &\leq (b-a)\left\{\left(\int_0^{1-\alpha} |t-\alpha\lambda|\, dt\right)^{1-\frac{1}{q}} \left(\int_0^{1-\alpha} |t-\alpha\lambda| \left|f'\left(tb+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\ &\left. + \left(\int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)|\, dt\right)^{1-\frac{1}{q}} \left(\int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)| \left|f'\left(tb+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right\} \\ &\leq (b-a)\left|f'(a)\right|\left|f'(b)\right|\left\{\int_0^{1-\alpha} |t-\alpha\lambda|\, dt + \int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)|\, dt\right\}. \end{aligned} \tag{2.4}$$

Hence, by simple computation

$$\int_0^{1-\alpha} |t-\alpha\lambda|\, dt = \begin{cases} \gamma_2, & \alpha\lambda \leq 1-\alpha \\ \gamma_1, & \alpha\lambda \geq 1-\alpha \end{cases}, \tag{2.5}$$

$$\gamma_1 = (1-\alpha)\left[\alpha\lambda - \frac{(1-\alpha)}{2}\right], \ \gamma_2 = (\alpha\lambda)^2 - \gamma_1 \ ,$$

$$\int_{1-\alpha}^{1} |t-1+\lambda(1-\alpha)|\,dt = \begin{cases} \upsilon_1, & 1-\lambda(1-\alpha) \le 1-\alpha \\ \upsilon_2, & 1-\lambda(1-\alpha) \ge 1-\alpha \end{cases}, \tag{2.6}$$

$$\begin{aligned} \upsilon_1 &= \frac{1-(1-\alpha)^2}{2} - \alpha\left[1-\lambda(1-\alpha)\right], \\ \upsilon_2 &= \frac{1+(1-\alpha)^2}{2} - (\lambda+1)(1-\alpha)\left[1-\lambda(1-\alpha)\right], \end{aligned}$$

$$\begin{aligned} \int_0^{1-\alpha} |t-\alpha\lambda| \left|f'(tb+(1-t)a)\right|^q dt &\le \int_0^{1-\alpha} |t-\alpha\lambda| \left|f'(b)\right|^q \left|f'(a)\right|^q dt \\ &= \left|f'(b)\right|^q \left|f'(a)\right|^q \int_0^{1-\alpha} |t-\alpha\lambda|\,dt \\ &= \begin{cases} \gamma_2 \left|f'(a)\right|^q \left|f'(b)\right|^q, & \alpha\lambda \le 1-\alpha \\ \gamma_1 \left|f'(a)\right|^q \left|f'(b)\right|^q, & \alpha\lambda \ge 1-\alpha \end{cases}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\int_{1-\alpha}^{1} |t-1+\lambda(1-\alpha)| \left|f'(tb+(1-t)a)\right|^q dt \\ &\le \int_{1-\alpha}^{1} |t-1+\lambda(1-\alpha)| \left|f'(b)\right|^q \left|f'(a)\right|^q dt \\ &= \left|f'(b)\right|^q \left|f'(a)\right|^q \int_{1-\alpha}^{1} |t-1+\lambda(1-\alpha)|\,dt \\ &= \begin{cases} \upsilon_1 \left|f'(b)\right|^q \left|f'(a)\right|^q, & 1-\lambda(1-\alpha) \le 1-\alpha \\ \upsilon_2 \left|f'(b)\right|^q \left|f'(a)\right|^q, & 1-\lambda(1-\alpha) \ge 1-\alpha \end{cases} \end{aligned} \tag{2.8}$$

Thus, using (2.5)-(2.8) in (2.4), we obtain the inequality (2.1). This completes the proof. □

Corollary 2.2 *Let the assumptions of Theorem* 2.1 *hold. Then for* $\alpha = \frac{1}{2}$ *and* $\lambda = \frac{1}{3}$, *from the inequality* (2.1) *we get the following Simpson type inequality*

$$\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx \right| \le \frac{5}{36}(b-a)\left|f'(a)\right|\left|f'(b)\right|. \tag{2.9}$$

Corollary 2.3 *Let the assumptions of Theorem* 2.1 *hold. Then for* $\alpha = \frac{1}{2}$ *and* $\lambda = 0$, *from the inequality* (2.1) *we get the following midpoint inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \le \frac{b-a}{4}\left|f'(a)\right|\left|f'(b)\right| \tag{2.10}$$

Remark 2.4 We note that the result obtained in Corollary 2.3 coincides with the result in [8].

Corollary 2.5 *Let the assumptions of Theorem 2.1 hold. Then for $\alpha = \frac{1}{2}$ and $\lambda = 1$, from the inequality (2.1) we get the following trapezoid inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left|f'(a)\right|\left|f'(b)\right|$$

Remark 2.6 We note that the result obtained in Corollary 2.5 coincides with the result in [8].

Using Lemma 1.4 we shall give another result for multiplicatively P-functions as follows.

Theorem 2.7 *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$ and $\alpha, \lambda \in [0,1]$. If $|f'|^q$ is multiplicatively P-function on $[a,b]$, $q > 1$, then the following inequality holds:*

$$\left|\lambda\left(\alpha f(a) + (1-\alpha) f(b)\right) + (1-\lambda) f(\alpha a + (1-\alpha) b) - \frac{1}{b-a}\int_a^b f(x)dx\right| \tag{2.11}$$

$$\leq (b-a)\left|f'(a)\right|\left|f'(b)\right|\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\begin{cases} (1-\alpha)^{\frac{1}{q}}\theta_1^{\frac{1}{p}} + \alpha^{\frac{1}{q}}\theta_2^{\frac{1}{p}}, & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ (1-\alpha)^{\frac{1}{q}}\theta_1^{\frac{1}{p}} + \alpha^{\frac{1}{q}}\theta_4^{\frac{1}{p}}, & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ (1-\alpha)^{\frac{1}{q}}\theta_3^{\frac{1}{p}} + \alpha^{\frac{1}{q}}\theta_2^{\frac{1}{p}}, & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases}$$

where

$$\begin{aligned} \theta_1 &= (\alpha\lambda)^{p+1} + (1-\alpha-\alpha\lambda)^{p+1}, & \theta_2 &= [\lambda(1-\alpha)]^{p+1} + [\alpha - \lambda(1-\alpha)]^{p+1} \\ \theta_3 &= (\alpha\lambda)^{p+1} - (1-\alpha-\alpha\lambda)^{p+1}, & \theta_4 &= [\lambda(1-\alpha)]^{p+1} - [\alpha - \lambda(1-\alpha)]^{p+1} \end{aligned} \tag{2.12}$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof From Lemma 1.4 and by Hölder's integral inequality, we have

$$\left|\lambda\left(\alpha f(a) + (1-\alpha) f(b)\right) + (1-\lambda) f(\alpha a + (1-\alpha) b) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq (b-a)\left[\int_0^{1-\alpha}|t-\alpha\lambda|\left|f'(tb+(1-t)a)\right|dt + \int_{1-\alpha}^{1}|t-1+\lambda(1-\alpha)|\left|f'(tb+(1-t)a)\right|dt\right]$$

$$\leq (b-a)\left\{\left(\int_0^{1-\alpha}|t-\alpha\lambda|^p dt\right)^{\frac{1}{p}}\left(\int_0^{1-\alpha}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\int_{1-\alpha}^{1}|t-1+\lambda(1-\alpha)|^p dt\right)^{\frac{1}{p}}\left(\int_{1-\alpha}^{1}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right\}$$

$$\leq (b-a)\left|f'(a)\right|\left|f'(b)\right|\left[(1-\alpha)^{\frac{1}{q}}\left(\int_0^{1-\alpha}|t-\alpha\lambda|^p dt\right)^{\frac{1}{p}}\right.$$

$$+\alpha^{\frac{1}{q}}\left(\int_{1-\alpha}^{1}|t-1+\lambda(1-\alpha)|^{p}\,dt\right)^{\frac{1}{p}}\Bigg]. \tag{2.13}$$

By simple computation

$$\int_{0}^{1-\alpha}|t-\alpha\lambda|^{p}\,dt=\begin{cases}\frac{(\alpha\lambda)^{p+1}+(1-\alpha-\alpha\lambda)^{p+1}}{p+1}, & \alpha\lambda\leq 1-\alpha\\ \frac{(\alpha\lambda)^{p+1}-(\alpha\lambda-1+\alpha)^{p+1}}{p+1}, & \alpha\lambda\geq 1-\alpha\end{cases}, \tag{2.14}$$

and

$$\int_{1-\alpha}^{1}|t-1+\lambda(1-\alpha)|^{p}\,dt=\begin{cases}\frac{[\lambda(1-\alpha)]^{p+1}+[\alpha-\lambda(1-\alpha)]^{p+1}}{p+1}, & 1-\alpha\leq 1-\lambda(1-\alpha)\\ \frac{[\lambda(1-\alpha)]^{p+1}-[\lambda(1-\alpha)-\alpha]^{p+1}}{p+1}, & 1-\alpha\geq 1-\lambda(1-\alpha)\end{cases}. \tag{2.15}$$

Thus, using (2.15) in (2.13), we obtain the inequality (2.11). This completes the proof. □

Corollary 2.8 *Let the assumptions of Theorem* 2.7 *hold. Then for* $\alpha=\frac{1}{2}$ *and* $\lambda=\frac{1}{3}$, *from the inequality* (2.11) *we get the following Simpson type inequality*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\leq\frac{b-a}{6}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}}\left|f'(a)\right|\left|f'(b)\right|. \tag{2.16}$$

Corollary 2.9 *Let the assumptions of Theorem* 2.7 *hold. Then for* $\alpha=\frac{1}{2}$ *and* $\lambda=0$, *from the inequality* (2.11) *we get the following midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\leq\frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left|f'(a)\right|\left|f'(b)\right|$$

Remark 2.10 Notice that the result obtained in Corollary 2.9 coincides with the result in [8].

Corollary 2.11 *Let the assumptions of Theorem* 2.7 *hold. Then for* $\alpha=\frac{1}{2}$ *and* $\lambda=1$, *from the inequality* (2.11) *we get the following trapezoid inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\leq\frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left|f'(a)\right|\left|f'(b)\right|.$$

Remark 2.12 Notice that the result obtained in Corollary 2.11 coincides with the result in [8].

§3. Some Applications for Special Means

Let us recall the following special means of the two nonnegative numbers a and b with $\alpha\in[0,1]$:

(1) The weighted arithmetic mean

$$A_{\alpha}=A_{\alpha}(a,b):=\alpha a+(1-\alpha)b,\ \ a,b\geq 0.$$

(2) The unweighted arithmetic mean

$$A = A(a,b) := \frac{a+b}{2},\ a,b \geq 0.$$

(3) The weighted geometric mean

$$G_\alpha = G_\alpha(a,b) := a^\alpha b^{1-\alpha},\ a,b > 0.$$

(4) The unweighted geometric mean

$$G = G(a,b) := \sqrt{ab},\ a,b > 0.$$

(5) The Logarithmic mean

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a},\quad a \neq b,\ a,b > 0.$$

(6) Then n-Logarithmic mean

$$L_n = L_n(a,b) := \left(\frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)}\right)^{\frac{1}{n}},\ n \in \mathbb{Z}\setminus\{-1,0\},\ a,b > 0,\ a \neq b.$$

Proposition 3.1 *Let $a,b \in \mathbb{R}$ with $0 < a < b$ and $n \in \mathbb{Z}^+ \cup \{0\}$. Then, for $\alpha, \lambda \in [0,1]$ and $q \geq 1$, we have the following inequality:*

$$\begin{aligned}
&\left|\lambda A_\alpha(a^n, b^n) + (1-\lambda) A_\alpha^n - L_n^n\right| \\
&\leq \begin{cases}
(b-a)\, n^2 (ab)^{n-1} [\gamma_2 + \upsilon_2] & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\
(b-a)\, n^2 (ab)^{n-1} [\gamma_2 + \upsilon_1] & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\
(b-a)\, n^2 (ab)^{n-1} [\gamma_1 + \upsilon_2] & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha)
\end{cases},
\end{aligned}$$

where $\gamma_1,\ \gamma_2,\ \upsilon_1,\ \upsilon_2$, numbers are defined as in (2.2) – (2.3).

Proof This assertion immediately follows from Theorem 2.1 in the case of $f(x) = x^n$, $x \in [1,\infty)$, $n \in \mathbb{Z}^+ \cup \{0\}$. □

Proposition 3.2 *Let $a,b \in \mathbb{R}$ with $0 < a < b$ and $n \in \mathbb{Z}^+ \cup \{0\}$. Then, for $\alpha, \lambda \in [0,1]$ and $q > 1$, we have the following inequality:*

$$\left|\lambda A_\alpha(a^n, b^n) + (1-\lambda) A_\alpha^n - L_n^n\right| \leq (b-a)\, n^2 G^{2n-2} \left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$

$$\times \begin{cases}
\left[(1-\alpha)^{\frac{1}{q}} \theta_1^{\frac{1}{p}} + \alpha^{\frac{1}{q}} \theta_2^{\frac{1}{p}}\right], & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\
\left[(1-\alpha)^{\frac{1}{q}} \theta_1^{\frac{1}{p}} + \alpha^{\frac{1}{q}} \theta_4^{\frac{1}{p}}\right], & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\
\left[(1-\alpha)^{\frac{1}{q}} \theta_3^{\frac{1}{p}} + \alpha^{\frac{1}{q}} \theta_2^{\frac{1}{p}}\right], & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha)
\end{cases},$$

where $\theta_1, \theta_2, \theta_3, \theta_4$ numbers are defined as in (2.12).

Proof This assertion immediately follows from Theorem 2.7 in the case of $f(x) = x^n$, $x \in [1,\infty)$, $n \in \mathbb{Z}^+ \cup \{0\}$. □

Proposition 3.3 *Let $a, b \in \mathbb{R}$ with $0 < a < b$. Then, for $\alpha, \lambda \in [0,1]$ and $q \geq 1$, we have the following inequality:*

$$\left| A_\lambda \left(A_\alpha \left(e^a, e^b \right), G_\alpha \left(e^a, e^b \right) \right) - L \left(e^a, e^b \right) \right|$$
$$\leq \begin{cases} (b-a)\, e^{2A} \left[\gamma_2 + \upsilon_2 \right] & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ (b-a)\, e^{2A} \left[\gamma_2 + \upsilon_1 \right] & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ (b-a)\, e^{2A} \left[\gamma_1 + \upsilon_2 \right] & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases},$$

where $\gamma_1,\ \gamma_2,\ \upsilon_1,\ \ \upsilon_2$, numbers are defined as in (2.2) – (2.3).

Proof The assertion follows from Theorem 2.1 in the case of $f(x) = e^x,\ x \in [0, \infty)$. □

Proposition 3.4 *Let $a, b \in \mathbb{R}$ with $0 < a < b$. Then, for $\alpha, \lambda \in [0,1]$ and $q > 1$, we have the following inequality:*

$$\left| A_\lambda \left(A_\alpha \left(e^a, e^b \right), G_\alpha \left(e^a, e^b \right) \right) - L \left(e^a, e^b \right) \right| \leq (b-a)\, e^{2A} \left(\frac{1}{p+1} \right)^{\frac{1}{p}}$$
$$\times \begin{cases} \left[(1-\alpha)^{\frac{1}{q}}\, \theta_1^{\frac{1}{p}} + \alpha^{\frac{1}{q}} \theta_2^{\frac{1}{p}} \right], & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ \left[(1-\alpha)^{\frac{1}{q}}\, \theta_1^{\frac{1}{p}} + \alpha^{\frac{1}{q}} \theta_4^{\frac{1}{p}} \right], & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ \left[(1-\alpha)^{\frac{1}{q}}\, \theta_3^{\frac{1}{p}} + \alpha^{\frac{1}{q}} \theta_2^{\frac{1}{p}} \right], & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases},$$

where $\theta_1, \theta_2, \theta_3, \theta_4$ numbers are defined as in (2.12).

Proof The assertion follows from Theorem 2.7 in the case of $f(x) = e^x,\ x \in [0, \infty)$. □

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C-Geometric Mean Labeling of Some Cycle Related Graphs

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Abstract: In a study of traffic, the labeling problems in graph theory can be used by considering the crowd at every junction as the weights of a vertex and expected average traffic in each street as the weight of the corresponding edge. If we assume the expected traffic at each street as the arithmetic mean of the weight of the end vertices, that causes mean labeling of the graph. When we consider a geometric mean instead of arithmetic mean in a large population of a city, the rate of growth of traffic in each street will be more accurate. The geometric mean labeling of graphs have been defined in which the edge labels may be assigned by either flooring function or ceiling function. In this, the readers will get some confusion in finding the edge labels which edge is assigned by flooring function and which edge is assigned by ceiling function. To avoid this confusion, we establish the C-Geometric mean labeling on graphs by considering the edge labels obtained only from the ceiling function. A C-Geometric mean labeling of a graph G with q edges, is an injective function from the vertex set of G to $\{1, 2, 3, \cdots, q+1\}$ such that the edge labels obtained from the ceiling function of geometric mean of the vertex labels of the end vertices of each edge, are all distinct and the set of edge labels is $\{2, 3, 4, \cdots, q+1\}$. A graph is said to be a C-Geometric mean graph if it admits a C-Geometric mean labeling. In this paper, we study the C-geometric meanness of some cycle related graphs such as cycle, union of a path and a cycle, union of two cycles, the graph $C_3 \times P_n$, corona of cycle, the graphs $P_{a,b}$, P_a^b and some chain graphs.

Key Words: Labeling, C-Geometric mean labeling, Smarandache $2k$-Geometric mean labeling, C-Geometric mean graph.

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§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V, E)$ be a graph with p vertices and q edges. For notations and terminology, we follow [4]. For a detailed survey on graph labeling we refer to [3].

Path on n vertices is denoted by P_n. $G \odot S_m$ is the graph obtained from G by attaching m pendant vertices at each vertex of G. Let G_1 and G_2 be any two graphs with p_1 and p_2 vertices respectively. Then the cartesian product $G_1 \times G_2$ has $p_1 p_2$ vertices which are $\{(u, v) : u \in G_1, v \in G_2\}$ and any two

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vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \times G_2$ if either $u_1 = u_2$ and v_1 and v_2 are adjacent in G_2 or u_1 and u_2 are adjacent in G_1 and $v_1 = v_2$.

Let u and v be two fixed vertices. We connect u and v by means of $b \geq 2$ internally disjoint paths of length $a \geq 2$ each. The resulting graph embedded in the plane is denoted by $P_{a,b}$. Let a and b be integers such that $a \geq 2$ and $b \geq 2$. Let $y_1, y_2, \cdots, y_a$ be the 'a' fixed vertices. We connect y_i and y_{i+1} by means of b internally disjoint paths of length $(i+1)$ for each i, $1 \leq i \leq a-1$. The resulting graph embedded in the plane is denoted by P_a^b.

Barrientos [1] defines a chain graph as one with blocks $B_1, B_2, B_3, \cdots, B_m$ such that for every i, B_i and B_{i+1} have a common vertex in such a way that the block cut point graph is a path. The chain graph $\widehat{G}(p_1, k_1, p_2, k_2, \cdots, k_{n-1}, p_n)$ is obtained from n cycles of length $p_1, p_2, p_3, \cdots, p_n$ and $(n-1)$ paths on $k_1, k_2, k_3, \cdots, k_{n-1}$ vertices respectively by identifying a cycle and a path at a vertex alternatively as follows. If the i^{th} cycle is of odd length, then its $\left(\frac{p_i+3}{2}\right)^{th}$ vertex is identified with a pendant vertex of the i^{th} path and if the i^{th} cycle is of even length, then its $\left(\frac{p_i+2}{2}\right)^{th}$ vertex is identified with a pendant vertex of the i^{th} path while the other pendant vertex of the i^{th} path is identified with the first vertex of the $(i+1)^{th}$ cycle. The chain graph $G^*(p_1, p_2, \cdots, p_n)$ is obtained from n cycles of length $p_1, p_2, \cdots, p_n$ by identifying consecutive cycles at a vertex as follows. If the i^{th} cycle is of odd length, then its $\left(\frac{p_i+3}{2}\right)^{th}$ vertex is identified with the first vertex of $(i+1)^{th}$ cycle and if the i^{th} cycle is of even length, then its $\left(\frac{p_i+2}{2}\right)^{th}$ vertex is identified with the first vertex of $(i+1)^{th}$ cycle. The graph $G'(p_1, p_2, \cdots, p_n)$ is obtained from n cycles of length $p_1, p_2, \cdots, p_n$ by identifying consecutive cycles at an edge as follows:

The $\left(\frac{p_j+3}{2}\right)^{th}$ *edge of* j^{th} *cycle is identified with the first edge of* $(j+1)^{th}$ *cycle when* j *is odd and the* $\left(\frac{p_j+1}{2}\right)^{th}$ *edge of* j^{th} *cycle is identified with the first edge of* $(j+1)^{th}$ *cycle when* j *is even.*

The study of graceful graphs and graceful labeling methods was first introduced by Rosa [5] and many authors are working in graph labeling [2,3]. Motivated by their methods, we introduce a new type of labeling called C-Geometric mean labeling. A function f is called a C-Geometric mean labeling of a graph G if $f : V(G) \to \{1, 2, 3, \cdots, q+1\}$ is injective and the induced function $f^* : E(G) \to \{2, 3, 4, \cdots, q+1\}$ defined as

$$f^*(uv) = \left\lceil \sqrt{f(u)f(v)} \right\rceil, \quad \text{for all} \quad uv \in E(G)$$

is bijective. Furthermore, if

$$f^*(uv) = \left\lceil \sqrt[2k]{f(u)^k f(v)^k} \right\rceil, \quad \text{for all} \quad uv \in E(G)$$

is bijective, where $k \geq 1$ is an integer, such a function f is called a Smarandache $2k$-Geometric mean labeling, and C-Geometric mean labeling of a graph G if $k = 1$. A graph that admits a C-Geometric mean labeling is called a C-Geometric mean graph.

In [6], S.Somasundaram et al. defined the geometric mean labeling as follows:

A graph $G = (V, E)$ *with* p *vertices and* q *edges is said to be a geometric mean graph if it is possible to label the vertices* $x \in V$ *with distinct labels* $f(x)$ *from* $1, 2, \cdots, q+1$ *in such way that when each edge* $e = uv$ *is labeled with* $f(uv) = \left\lfloor \sqrt{f(u)f(v)} \right\rfloor$ *or* $\left\lceil \sqrt{f(u)f(v)} \right\rceil$ *then the edge labels are distinct.*

In the above definition, the readers will get some confusion in finding the edge labels which edge is assigned by flooring function and which edge is assigned by ceiling function.

In [6], the authors have given a geometric mean labeling of the graph $C_5 \cup C_7$ as in the Figure 1.

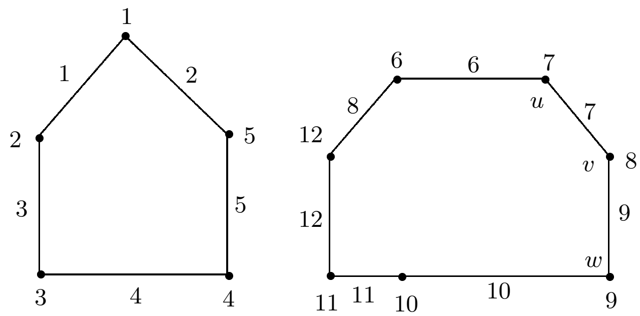

Figure 1 A geometric mean labeling of $C_5 \cup C_7$.

From the above figure, for the edge uv, they have used flooring function $\left\lfloor \sqrt{f(u)f(v)} \right\rfloor$ and for the edge vw, they have used ceiling function $\left\lceil \sqrt{f(u)f(v)} \right\rceil$ for fulfilling their requirement. To avoid the confusion of assigning the edge labels in their definition, we just consider the ceiling function $\left\lceil \sqrt{f(u)f(v)} \right\rceil$ for our discussion. Based on our definition, the C-Geometric mean labeling of the same graph $C_5 \cup C_7$ is given in Figure 2.

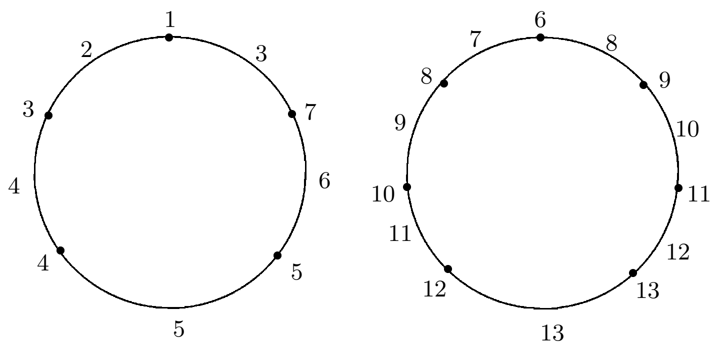

Figure 2 A C-Geometric mean labeling of $C_5 \cup C_7$

In this paper, we have discussed the C-Geometric mean labeling of the cycle for $n \geq 4$, union of any two cycles C_m and C_n, union of the cycle C_m and a path P_n, the graph $C_3 \times P_n$, corona of cycle, the graphs $P_{a,b}$, P_a^b and some chain graphs.

§2. Main Results

Theorem 2.1 *A graph C_n is a C-Geometric mean graph only if $n \geq 4$.*

Proof The proof is divided into 2 cases following.

Case 1. $n \geq 4$.

Let $v_1, v_2, \cdots, v_n$ be the vertices of C_n. Define $f : V(C_n) \to \{1, 2, 3, \cdots, n+1\}$ as follows:

$$f(v_i) = \begin{cases} 2i-1, & 1 \leq i \leq 2, \\ 2i-2, & 3 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1, \\ n+1, & i = \left\lfloor \frac{n}{2} \right\rfloor + 2, \\ 2n+5-2i, & \left\lfloor \frac{n}{2} \right\rfloor + 3 \leq i \leq n. \end{cases}$$

Then, the induced edge labeling is obtained as follows:

$$f^*(v_iv_{i+1}) = \begin{cases} 2i, & 1 \le i \le 2, \\ 2i-1, & 3 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1, \\ n+1, & i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is odd}, \\ n, & i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is even}, \\ 2n+4-2i, & 3 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 3 \le i \le n-1 \end{cases}$$

and $f^*(v_nv_1) = 3$.

Hence, f is a C-Geometric mean labeling of the cycle C_n. Thus the cycle C_n is a C-Geometric mean graph for $n \ge 4$.

Case 2. $n = 3$.

Let v_1, v_2 and v_3 be the vertices of C_3. To get the edge label $q+1, q$ and $q+1$ should be the vertex labels for two of the vertices of C_3, say $v_1 = q = 3$ and $v_2 = q+1 = 4$. Also to obtain the edge label 2, 1 is to be a vertex label of a vertex of C_3, say $v_3 = 1$. Since the edge labels of the edges v_1v_3 and v_2v_3 are one and the same. Hence C_3 is not a C-Geometric mean graph. □

Theorem 2.2 *A union of two cycles C_m and C_n is a C-Geometric mean graph if $m \ge 3$ and $n \ge 3$.*

Proof Let $u_1, u_2, \cdots, u_m$ and $v_1, v_2, \cdots, v_n$ be the vertices of the cycles C_m and C_n respectively.

Case 1. $m \ge 4$ or $n \ge 4$.

Define $f : V(C_m \cup C_n) \to \{1, 2, 3, \cdots, m+n+1\}$ as follows:

$$f(u_i) = \begin{cases} i, & 1 \le i \le \left\lceil \sqrt{m+2} \right\rceil - 2, \\ i+1, & \left\lceil \sqrt{m+2} \right\rceil - 1 \le i \le m-1, \end{cases}$$

$$f(u_m) = m+2,$$

$$f(v_i) = \begin{cases} m-1+2i, & 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor + 1, \\ m+2n+4-2i, & \left\lfloor \frac{n}{2} \right\rfloor + 2 \le i \le n. \end{cases}$$

Then, the induced edge labeling is known as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} i+1, & 1 \le i \le \left\lceil \sqrt{m+2} \right\rceil - 2, \\ i+2, & \left\lceil \sqrt{m+2} \right\rceil - 1 \le i \le m-1, \end{cases}$$

$$f^*(u_1, u_m) = \left\lceil \sqrt{m+2} \right\rceil,$$

$$f^*(v_iv_{i+1}) = \begin{cases} m+2i, & 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor, \\ m+n+1, & i = \left\lfloor \frac{n}{2} \right\rfloor + 1, \\ m+2n+3-2i, & \left\lfloor \frac{n}{2} \right\rfloor + 2 \le i \le n-1 \end{cases}$$

and $f^*(v_1v_n) = m+3$.

Hence, f is a C-Geometric mean labeling of the graph $C_m \cup C_n$. Thus the graph $C_m \cup C_n$ is a C-Geometric mean graph, for $m \ge 4$ or $n \ge 4$.

Case 2. $m = 3$ and $n = 3$.

A C-Geometric mean labeling of $C_3 \cup C_3$ is shown in Figure 3.

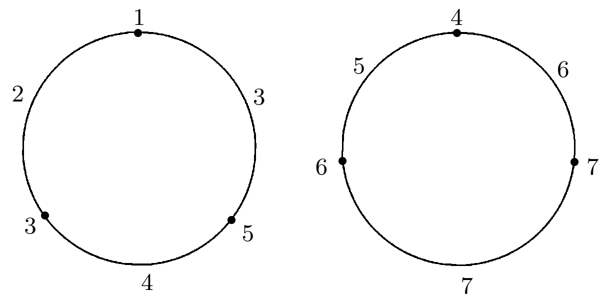

Figure 3 A C-Geometric mean labeling of $C_3 \cup C_3$.

This completes the proof. □

Theorem 2.3 *A graph $C_m \cup P_n$ is a C-Geometric mean graph if $m \geq 3$ and $n \geq 2$.*

Proof Let $u_1, u_2, \cdots, u_m$ and $v_1, v_2, \cdots, v_n$ be the vertices of the cycle C_m and the path P_n respectively.

Define $f : V(C_m \cup P_n) \to \{1, 2, 3, \cdots, m+n\}$ as follows:

$$f(u_i) = \begin{cases} n + 2i - 2, & 1 \leq i \leq \lfloor \frac{m}{2} \rfloor + 1, \\ n + 2m + 3 - 2i, & \lfloor \frac{m}{2} \rfloor + 2 \leq i \leq m, \end{cases}$$

$$f(v_i) = i, \text{ for } 1 \leq i \leq n-1 \text{ and}$$

$$f(v_n) = n + 1.$$

Then, the induced edge labeling is obtained as follows:

$$f^*(u_i u_{i+1}) = \begin{cases} n - 1 + 2i, & 1 \leq i \leq \lfloor \frac{m}{2} \rfloor, \\ m + n, & i = \lfloor \frac{m}{2} \rfloor + 1, \\ n + 2m + 2 - 2i, & \lfloor \frac{m}{2} \rfloor + 2 \leq i \leq m - 1, \end{cases}$$

$$f^*(u_1 u_m) = n + 2 \text{ and}$$

$$f^*(v_i v_{i+1}) = i + 1, \text{ for } 1 \leq i \leq n - 1.$$

Hence, f is a C-Geometric mean labeling of the graph $C_m \cup P_n$. Thus the graph $C_m \cup P_n$ is a C-Geometric mean graph, for $m \geq 3$ and $n \geq 2$. □

Theorem 2.4 *A graph $C_3 \times P_n$ is a C-Geometric mean graph if $n \geq 4$.*

Proof Let $V(C_3 \times P_n) = \{v_1^{(i)}, v_2^{(i)}, v_3^{(i)}; 1 \leq i \leq n\}$ be the vertex set of $C_3 \times P_n$ and $E(C_3 \times P_n) = \{v_1^{(i)} v_2^{(i)}, v_2^{(i)} v_3^{(i)}, v_1^{(i)} v_3^{(i)}; 1 \leq i \leq n\} \cup \{v_1^{(i)} v_1^{(i+1)}, v_2^{(i)} v_2^{(i+1)}, v_3^{(i)} v_3^{(i+1)}; 1 \leq i \leq n-1\}$ be the edge set of $C_3 \times P_n$.

Define $f : V(C_3 \times P_n) \to \{1, 2, 3, \cdots, 6n-2\}$ as follows

$$f(v_1^{(j)}) = \begin{cases} 9j-8, & 1 \le j \le 2, \\ 8j-11, & 3 \le j \le 4, \end{cases}$$

$$f(v_2^{(j)}) = \begin{cases} 6j-3, & 1 \le j \le 2, \\ 2j+11, & 3 \le j \le 4, \end{cases}$$

$$f(v_3^{(j)}) = \begin{cases} 5+j, & 1 \le j \le 2, \\ 7j-6, & 3 \le j \le 4 \end{cases}$$

and $f(v_i^{(j)}) = f(v_i^{(j-3)}) + 18$ for $1 \le i \le 3$ and $5 \le j \le n$. Then, the induced edge labeling is obtained as follows:

$$f^*(v_1^{(j)} v_2^{(j)}) = \begin{cases} 2, & j = 1, \\ 5j, & 2 \le j \le 3, \\ f^*(v_1^{(j-3)} v_2^{(j-3)}) + 18, & 4 \le j \le n, \end{cases}$$

$$f^*(v_2^{(j)} v_3^{(j)}) = \begin{cases} 3j+2, & 1 \le j \le 2, \\ 5j+1, & 3 \le j \le 4, \\ f^*(v_2^{(j-3)} v_3^{(j-3)}) + 18, & 5 \le j \le n, \end{cases}$$

$$f^*(v_1^{(j)} v_3^{(j)}) = \begin{cases} 6j-3, & 1 \le j \le 2, \\ 8j-10, & 3 \le j \le 4, \\ f^*(v_1^{(j-3)} v_3^{(j-3)}) + 18, & 5 \le j \le n, \end{cases}$$

$$f^*(v_1^{(j)} v_1^{(j+1)}) = \begin{cases} 8j-4, & 1 \le j \le 2 \\ 8j-7, & 3 \le j \le 4, \\ f^*(v_1^{(j-3)} v_1^{(j-2)}) + 18, & 5 \le j \le n-1, \end{cases}$$

$$f^*(v_2^{(j)} v_2^{(j+1)}) = \begin{cases} 6, & j = 1 \\ 5j+3, & 2 \le j \le 4, \\ f^*(v_2^{(j-3)} v_2^{(j-2)}) + 18, & 5 \le j \le n-1, \end{cases}$$

$$f^*(v_3^{(j)} v_3^{(j+1)}) = \begin{cases} 4j+3, & 1 \le j \le 2, \\ 5j+4, & 3 \le j \le 4 \\ f^*(v_3^{(j-3)} v_3^{(j-2)}) + 18, & 5 \le j \le n-1. \end{cases}$$

Hence f is a C-Geometric mean labeling of $C_3 \times P_n$. Thus the graph $C_3 \times P_n$ is a C-Geometric mean graph, for $n \ge 4$. □

Theorem 2.5 *A graph $C_n \odot S_m$ is a C-Geometric mean graph if $n \ge 3$ and $m \le 2$.*

Proof Let $u_1, u_2, \cdots, u_n$ be the vertices of the cycle C_n and let $v_0^{(i)}, v_1^{(i)}, \cdots, v_m^{(i)}$ be the vertices of the star graph S_m such that $v_0^{(i)}$ is the central vertex of S_m, for $1 \leq i \leq n$.

Case 1. $m = 1$.

Subcase 1.1 $\left\lceil \sqrt{2(2n+1)} \right\rceil$ is odd and $n \geq 5$.

Define $f : V(C_n \odot S_1) \to \{1, 2, 3, \cdots, 2n+1\}$ as follows:

$$f(u_i) = \begin{cases} 2i, & 1 \leq i \leq \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor, \\ 2i+1, & \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor + 1 \leq i \leq i \leq n, \end{cases}$$

$$f(v_1^{(i)}) = \begin{cases} 2i-1, & 1 \leq i \leq \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor, \\ 2i, & \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor + 1 \leq i \leq i \leq n. \end{cases}$$

Then, the induced edge labeling is obtained as follows:

$$f^*(u_i u_{i+1}) = \begin{cases} 2i+1, & 1 \leq i \leq \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor - 1, \\ 2i+2, & \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor + 1 \leq i \leq i \leq n-1, \end{cases}$$

$$f^*(u_1 u_n) = \left\lceil \sqrt{2(2n+1)} \right\rceil,$$

$$f^*(u_i v_1^{(i)}) = \begin{cases} 2i, & 1 \leq i \leq \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor, \\ 2i+1, & \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor + 1 \leq i \leq n. \end{cases}$$

Subcase 1.2 $\left\lceil \sqrt{2(2n+1)} \right\rceil$ is even.

Define $f : V(C_n \odot S_2) \to \{1, 2, 3, \cdots, 2n+1\}$ as follows:

$$f(u_i) = \begin{cases} 2i, & 1 \leq i \leq \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor - 2, \\ 2i-1, & i = \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor - 1, \\ 2i+1, & \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor \leq i \leq i \leq n, \end{cases}$$

$$f(v_1^{(i)}) = \begin{cases} 2i-1, & 1 \leq i \leq \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor - 2, \\ 2i, & \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor - 1 \leq i \leq i \leq n. \end{cases}$$

Then, the induced edge labeling is obtained as follows:

$$f^*(u_i u_{i+1}) = \begin{cases} 2i+1, & 1 \leq i \leq \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor - 1, \\ 2i+2, & \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor \leq i \leq i \leq n-1, \end{cases}$$

$$f^*(u_1u_n) = \left\lceil \sqrt{2(2n+1)} \right\rceil$$

$$\text{and } f^*(u_iv_1^{(i)}) = \begin{cases} 2i, & 1 \leq i \leq \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor - 1, \\ 2i+1, & \left\lfloor \frac{\sqrt{2(2n+1)}}{2} \right\rfloor \leq i \leq n. \end{cases}$$

Hence, the graph $C_n \odot S_1$, for $n \geq 4$ admits a C-Geometric mean labeling. For $n = 3$, a C-Geometric mean labeling of $C_3 \odot S_1$ is shown in Figure 4.

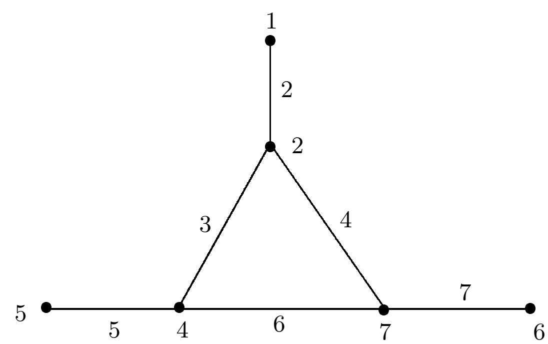

Figure 4 A C-Geometric mean labeling of $C_3 \odot S_1$.

Case 2. $m = 2$.

Subcase 2.1 $\left\lceil \sqrt{6n} \right\rceil \equiv 0 (mod\ 3)$.

Define $f : V(C_n \odot S_2) \to \{1, 2, 3, \cdots, 3n+1\}$ as follows:

$$f(u_i) = \begin{cases} 3i-1, & 1 \leq i \leq \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor - 1, \\ 3i, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor \leq i \leq n, \end{cases}$$

$$f(v_1^{(i)}) = \begin{cases} 3i-2, & 1 \leq i \leq \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i-1, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \leq i \leq n, \end{cases}$$

$$f(v_2^{(i)}) = \begin{cases} 3i, & 1 \leq i \leq \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor - 1, \\ 3i+1, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor \leq i \leq n. \end{cases}$$

Then, the induced edge labeling is obtained as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} 3i+1, & 1 \leq i \leq \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor - 1, \\ 3i+2, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor \leq i \leq n-1, \end{cases}$$

$$f^*(u_nu_1) = \left\lceil \sqrt{6n} \right\rceil,$$

$$f^*(u_iv_1^{(i)}) = \begin{cases} 3i-1, & 1 \leq i \leq \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \leq i \leq n, \end{cases}$$

$$f^*(u_iv_2^{(i)}) = \begin{cases} 3i, & 1 \leq i \leq \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor - 1, \\ 3i+1, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor \leq i \leq n. \end{cases}$$

Subcase 2.2 $\lceil\sqrt{6n}\rceil \equiv 1(mod\ 3)$.

Define $f : V(C_n \odot S_2) \to \{1, 2, 3, \cdots, 3n+1\}$ as follows:

$$f(u_i) = \begin{cases} 3i-1, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i+1, & i = \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1, \\ 3i, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 2 \le i \le n, \end{cases}$$

$$f(v_1^{(i)}) = \begin{cases} 3i-2, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i-1, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n, \end{cases}$$

$$f(v_2^{(i)}) = \begin{cases} 3i, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1, \\ 3i+1, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 2 \le i \le n. \end{cases}$$

Then, the induced edge labeling is obtained as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} 3i+1, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor - 1, \\ 3i+2, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor \le i \le n-1, \end{cases}$$

$$f^*(u_nu_1) = \left\lceil \sqrt{6n} \right\rceil,$$

$$f^*(u_iv_1^{(i)}) = \begin{cases} 3i-1, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n, \end{cases}$$

$$f^*(u_iv_2^{(i)}) = \begin{cases} 3i, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i+1, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n. \end{cases}$$

Subcase 2.3 $\lceil\sqrt{6n}\rceil \equiv 2(mod\ 3)$.

Define $f : V(C_n \odot S_2) \to \{1, 2, 3, \cdots, 3n+1\}$ as follows:

$$f(u_i) = \begin{cases} 3i-1, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n, \end{cases}$$

$$f(v_1^{(i)}) = \begin{cases} 3i-2, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i-1, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n, \end{cases}$$

$$f(v_2^{(i)}) = \begin{cases} 3i, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i+1, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n. \end{cases}$$

Then, the induced edge labeling is obtained as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} 3i+1, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i+2, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n-1, \end{cases}$$

$$f^*(u_nu_1) = \left\lceil \sqrt{6n} \right\rceil,$$

$$f^*(u_iv_1^{(i)}) = \begin{cases} 3i-1, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n, \end{cases}$$

$$f^*(u_iv_2^{(i)}) = \begin{cases} 3i, & 1 \le i \le \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor, \\ 3i+1, & \left\lfloor \frac{\sqrt{6n}}{3} \right\rfloor + 1 \le i \le n. \end{cases}$$

Hence, the graph $C_n \odot S_2$, for $n \ge 3$ admits a C-Geometric mean labeling. Thus the graph $C_n \odot S_m$ is a C-Geometric mean graph, for $n \ge 3$ and $m \le 2$. □

Theorem 2.6 *A graph $\widehat{G}(p_1, m_1, p_2, m_2, \cdots, m_{n-1}, p_n)$ is a C-Geometric mean graph if $p_1 \ne 3$.*

Proof Let $\{v_i^{(j)}; 1 \le j \le n, 1 \le i \le p_j\}$ and $\{u_i^{(j)}; 1 \le j \le n-1, 1 \le i \le m_j\}$ be the n number of cycles and $(n-1)$ number of paths respectively. For $1 \le j \le n-1$, the j^{th} cycle and j^{th} path are identified by a vertex $v^{(j)}_{\frac{p_j+2}{2}}$ and $u_1^{(j)}$ while p_j is even and $v^{(j)}_{\frac{p_j+3}{2}}$ and $u_1^{(j)}$ while p_j is odd and the j^{th} path and $(j+1)^{th}$ cycle are identified by a vertex $u_{m_j}^{(j)}$ and $v_1^{(j+1)}$.

Define $f : V(\widehat{G}(p_1, m_1, p_2, m_2, \cdots, m_{n-1}, p_n)) \to \left\{1, 2, 3, \cdots, \sum\limits_{j=1}^{n-1}(p_j + m_j) \ + p_n - n + 2\right\}$ as follows:

If p_1 is odd and $p_1 \ne 3$,

$$f\left(v_i^{(1)}\right) = \begin{cases} 2i-1, & 1 \le i \le 2, \\ 2i-2, & 3 \le i \le \left\lfloor \frac{p_1}{2} \right\rfloor + 2, \\ 2p_1 + 5 - 2i, & \left\lfloor \frac{p_1}{2} \right\rfloor + 3 \le i \le p_1. \end{cases}$$

and if p_1 is even,

$$f\left(v_i^{(1)}\right) = \begin{cases} 3, & j = 1, \\ 2i, & 2 \le j \le \left\lfloor \frac{p_1}{2} \right\rfloor, \\ 2p_1 + 3 - 2i, & \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \le j \le p_1 - 1. \end{cases}$$

$f\left(v_{p_1}^{(1)}\right) = 1$, $f\left(u_i^{(1)}\right) = p_1 + i$, for $2 \le i \le m_1$. For $2 \le j \le n$,

$$f(v_i^{(j)}) = \begin{cases} \sum\limits_{k=1}^{j-1}(p_k + m_k) + 2i - j, & 2 \le i \le \left\lfloor \frac{p_j}{2} \right\rfloor + 1, \\ \sum\limits_{k=1}^{j-1}(p_k + m_k) + 2i - j - 1, & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \text{ and } p_j \text{ is odd}, \\ \sum\limits_{k=1}^{j-1}(p_k + m_k) + 2i - j - 3, & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \text{ and } p_j \text{ is even}, \\ \sum\limits_{k=1}^{j-1}(p_k + m_k) + 2p_j - 2i - j - 5, & \left\lfloor \frac{p_j}{2} \right\rfloor + 3 \le i \le p_j \end{cases}$$

and for $3 \le j \le n$,

$$f(u_i^{(j-1)}) = \sum_{k=1}^{j-2}(p_k + m_k) + p_{j-1} + i + 2 - j, \text{ for } 2 \le i \le m_{j-1}.$$

Then the induced edge labeling is obtained as follows:

If p_1 is odd and $p_1 \neq 3$,

$$f^*(v_i^{(1)}v_{i+1}^{(1)}) = \begin{cases} 2i, & 1 \leq i \leq 2, \\ 2i-1, & 3 \leq i \leq \lfloor \frac{p_1}{2} \rfloor + 1, \\ 2p_1 + 4 - 2i, & \lfloor \frac{p_1}{2} \rfloor + 2 \leq i \leq p_1 - 1, \end{cases}$$

$$f^*(v_{p_1}^{(1)}v_1^{(1)}) = 3.$$

and if p_1 is even,

$$f^*(v_i^{(1)}v_{i+1}^{(1)}) = \begin{cases} 4, & j = 1, \\ 2i+1, & 2 \leq i \leq \lfloor \frac{p_1}{2} \rfloor, \\ 2p_1 + 2 - 2i, & \lfloor \frac{p_1}{2} \rfloor + 1 \leq i \leq p_1 - 2, \end{cases}$$

$$f^*(v_{p_1-1}^{(1)}v_{p_1}^{(1)}) = 3,$$
$$f^*(v_{p_1}^{(1)}v_1^{(1)}) = 2,$$
$$f^*(u_i^{(1)}u_{i+1}^{(1)}) = p_1 + i, \quad 1 \leq i \leq m_1 - 1$$

and for $2 \leq j \leq n$,

$$f^*(v_i^{(j)}v_{i+1}^{(j)}) = \begin{cases} \sum\limits_{k=1}^{j-1}(p_k + m_k) + 2i - j + 1, & 1 \leq i \leq \lfloor \frac{p_j}{2} \rfloor, \\ \sum\limits_{k=1}^{j-1}(p_k + m_k) + 2i - j + 1, & i = \lfloor \frac{p_j}{2} \rfloor + 1 \text{ and } p_j \text{ is odd}, \\ \sum\limits_{k=1}^{j-1}(p_k + m_k) + 2p_j - 2i - j + 4, & i = \lfloor \frac{p_j}{2} \rfloor + 1 \text{ and } p_j \text{ is even}, \\ \sum\limits_{k=1}^{j-1}(p_k + m_k) + 2p_j - 2i - j + 4, & \lfloor \frac{p_j}{2} \rfloor + 2 \leq i \leq p_j - 1, \end{cases}$$

$$f^*(v_{p_j}^{(j)}v_1^{(j)}) = \sum_{k=1}^{j-1}(p_k + m_k) - j + 4$$

and

$$f^*(u_i^{(j-1)}u_{i+1}^{(j-1)}) = \sum_{k=1}^{j-2}(p_k + mk) + p_{j-1} + i + 3 - j, \text{ for } 1 \leq i \leq m_{j-1} - 1 \text{ and } 3 \leq j \leq n.$$

Hence, f is a C-Geometric mean labeling of $\widehat{G}(p_1, m_1, p_2, m_2 \cdots, m_{n-1}, p_n)$. Thus the graph $\widehat{G}(p_1, m_1, p_2, m_2 \cdots, m_{n-1}, p_n)$ is a C-Geometric mean graph, for $p_1 \neq 3$. □

Corollary 2.7 *A graph $G^*(p_1, p_2, \cdots, p_n)$ is a C-Geometric mean graph if $p_1 \neq 3$.*

Theorem 2.8 *A graph $G'(p_1, p_2, \cdots, p_n)$ is a C-Geometric mean graph if all p_j's are odd and $p_1 \neq 3$ or all p_j's $1 \leq j \leq n$ are even.*

Proof Let $\{v_i^{(j)}; 1 \leq j \leq n, 1 \leq i \leq p_j\}$ be the vertices of the n number of cycles.

Case 1. p_j is odd and $p_1 \neq 3$ for $1 \leq j \leq n$.

For $1 \leq j \leq n-1$, the j^{th} and $(j+1)^{th}$ cycles are identified by the edges $v_{\frac{p_j+1}{2}}^{(j)}v_{\frac{p_j+3}{2}}^{(j)}$ and $v_1^{(j+1)}v_{p_{j+1}}^{(j+1)}$ while j is odd and $v_{\frac{p_j-1}{2}}^{(j)}v_{\frac{p_j+1}{2}}^{(j)}$ and $v_1^{(j+1)}v_{p_{j+1}}^{(j+1)}$ while j is even.

Define $f : V(G'(p_1, p_2, \cdots, p_n)) \to \left\{1, 2, 3, \cdots, \sum\limits_{j=1}^{n} p_j - n + 2\right\}$ as follows:

$$f(v_i^{(1)}) = \begin{cases} 3, & i = 1, \\ 2i, & 2 \leq i \leq \lfloor \frac{p_1}{2} \rfloor + 1, \\ 2p_1 + 3 - 2i, & \lfloor \frac{p_1}{2} \rfloor + 2 \leq i \leq p_1 - 1, \end{cases}$$
$$f(v_{p_1}^{(1)}) = 1$$

and for $2 \leq j \leq n$,

$$f(v_i^{(j)}) = \begin{cases} \sum\limits_{k=1}^{j-1} p_k - j + 2i + 2, & 2 \leq i \leq \lfloor \frac{p_j}{2} \rfloor \text{ and } j \text{ is even}, \\ \sum\limits_{k=1}^{j-1} p_k + 2p_j + 3 - j - 2i, & \lfloor \frac{p_j}{2} \rfloor + 1 \leq i \leq p_j - 1 \text{ and } j \text{ even}, \\ \sum\limits_{k=1}^{j-1} p_k - j + 2i + 1, & 2 \leq i \leq \lfloor \frac{p_j}{2} \rfloor + 1 \text{ and } j \text{ is odd}, \\ \sum\limits_{k=1}^{j-1} p_k + 2p_j + 4 - j - 2i, & \lfloor \frac{p_j}{2} \rfloor + 2 \leq i \leq p_j - 1 \text{ and } j \text{ odd}. \end{cases}$$

The induced edge labeling is obtained as follows:

$$f^*(v_i^{(1)} v_{i+1}^{(1)}) = \begin{cases} 4, & i = 1, \\ 2i + 1, & 2 \leq i \leq \lfloor \frac{p_1}{2} \rfloor, \\ 2p_1 + 2 - 2i & \lfloor \frac{p_1}{2} \rfloor + 1 \leq i \leq p_1 - 2, \end{cases}$$
$$f^*(v_{p_1-1}^{(1)} v_{p_1}^{(1)}) = 3, \quad f^*(v_{p_1}^{(1)} v_1^{(1)}) = 2$$

and for $2 \leq j \leq n$,

$$f^*(v_i^{(j)} v_{i+1}^{(j)}) = \begin{cases} \sum\limits_{k=1}^{j-1} p_k - j + 2i + 3, & 1 \leq i \leq \lfloor \frac{p_j}{2} \rfloor \text{ and } j \text{ is even}, \\ \sum\limits_{k=1}^{j-1} p_k + 2p_j + 2 - j - 2i, & \lfloor \frac{p_j}{2} \rfloor + 1 \leq i \leq p_j - 1 \text{ and } j \text{ even}, \\ \sum\limits_{k=1}^{j-1} p_k - j + 2i + 2, & 1 \leq i \leq \lfloor \frac{p_j}{2} \rfloor \text{ and } j \text{ is odd}, \\ \sum\limits_{k=1}^{j-1} p_k + 2p_j + 3 - j - 2i, & \lfloor \frac{p_j}{2} \rfloor + 1 \leq i \leq p_j - 1 \text{ and } j \text{ odd}. \end{cases}$$

Case 2. p_j is even for $1 \leq j \leq n$.

For $1 \leq j \leq n-1$, the j^{th} and $(j+1)^{th}$ cycles are identified by the edges $v_{\frac{p_j}{2}}^{(j)} v_{\frac{p_j+2}{2}}^{(j)}$ and $v_1^{(j+1)} v_{p_{j+1}}^{(j+1)}$.

Define $f : V(G'(p_1, p_2, \cdots, p_n)) \to \left\{1, 2, 3, \cdots, \sum\limits_{j=1}^{n} p_j - n + 2\right\}$ as follows:

$$f(v_i^{(1)}) = \begin{cases} 3, & i = 1, \\ 2i, & 2 \leq i \leq \lfloor \frac{p_1}{2} \rfloor, \\ 2p_1 + 3 - 2i, & \lfloor \frac{p_1}{2} \rfloor + 1 \leq i \leq p_1 - 1, \end{cases}$$
$$f(v_{p_1}^{(1)}) = 1$$

and for $2 \leq j \leq n$,

$$f(v_i^{(j)}) = \begin{cases} \sum\limits_{k=1}^{j-1} p_k - j + 2i + 1, & 2 \leq i \leq \lfloor \frac{p_j}{2} \rfloor, \\ \sum\limits_{k=1}^{j-1} p_k + 2p_j + 4 - j - 2i, & \lfloor \frac{p_j}{2} \rfloor + 1 \leq i \leq p_j - 1. \end{cases}$$

The induced edge labeling is obtained as follows:

$$f^*(v_i^{(1)} v_{i+1}^{(1)}) = \begin{cases} 4, & i = 1, \\ 2i + 1, & 2 \leq i \leq \lfloor \frac{p_1}{2} \rfloor, \\ 2p_1 + 2 - 2i, & \lfloor \frac{p_1}{2} \rfloor + 1 \leq i \leq p_1 - 2, \end{cases}$$

$$f^*(v_{p_1-1}^{(1)} v_{p_1}^{(1)}) = 3,$$

$$f^*(v_{p_1}^{(1)} v_1^{(1)}) = 2$$

and for $2 \leq j \leq n$,

$$f^*(v_i^{(j)} v_{i+1}^{(j)}) = \begin{cases} \sum\limits_{k=1}^{j-1} p_k - j + 2i + 2, & 1 \leq i \leq \lfloor \frac{p_j}{2} \rfloor, \\ \sum\limits_{k=1}^{j-1} p_k + 2p_j + 3 - j - 2i, & \lfloor \frac{p_j}{2} \rfloor + 1 \leq i \leq p_j - 1. \end{cases}$$

Hence, f is a C-Geometric mean labeling of $G'(p_1, p_2, \ldots, p_n)$. Thus the graph $G'(p_1, p_2, \ldots, p_n)$ is a C-Geometric mean graph, for $p_1 \neq 3$. □

Theorem 2.9 *A graph $P_{a,b}$ is a C-Geometric mean graph if $b \leq 4$ and $a \geq 2$.*

Proof Let $v_0^{(i)}, v_1^{(i)}, v_2^{(i)}, \cdots, v_a^{(i)}$ be the vertices of the i^{th} copy of the path of length 'a' where $i = 1, 2, \cdots, b$, $v_0^{(i)} = u$ and $v_a^{(i)} = v$, for all i. Clearly, $|V(P_{a,b})| = ab - b + 2$ and $|E(P_{a,b})| = ab$. Consider a graph $P_{a,b}$ with $a \geq 2$.

Case 1. $b = 2$.

Notice that $P_{a,2}$ is a cycle of length more than 3. By Theorem 2.1, it admits a C-Geometric mean labeling.

Case 2. $b = 3$.

Define $f : V(P_{a,3}) \to \{1, 2, 3, \cdots, 3a + 1\}$ as follows:

$$f(u) = a + 1,$$

$$f(v) = 3a + 1,$$

$$f(v_{a-j}^{(1)}) = \begin{cases} j, & 1 \leq j \leq \lceil \sqrt{3a+1} \rceil - 2, \\ j + 1, & \lceil \sqrt{3a+1} \rceil - 1 \leq j \leq a - 1, \end{cases}$$

$$f(v_j^{(i)}) = a + i - 1 + 2j, \text{ for } 2 \leq i \leq 3,\ 1 \leq j \leq a - 1.$$

Then, the induced edge labeling is obtained as follows:

$$f^*(v_{a-j}^{(1)}v_{a-j-1}^{(1)}) = \begin{cases} j+1, & 1 \le j \le \left\lceil \sqrt{3a+1} \right\rceil - 2, \\ j+2, & \left\lceil \sqrt{3a+1} \right\rceil - 1 \le j \le a-2, \end{cases}$$

$$\begin{aligned} f^*(vv_{a-1}^{(1)}) &= \left\lceil \sqrt{3a+1} \right\rceil, \\ f^*(uv_1^{(1)}) &= a+1, \end{aligned}$$

$$\begin{aligned} f^*(vv_{a-1}^{(i)}) &= 3a-2+i, \quad \text{for } 2 \le i \le 3, \\ f^*(uv_1^{(i)}) &= a+i, \quad \text{for } 2 \le i \le 3, \\ f^*(v_j^{(i)}v_{j+1}^{(i)}) &= a+i+2j, \quad \text{for } 2 \le i \le 3 \text{ and } 1 \le j \le a-2. \end{aligned}$$

Case 3. $b = 4$.

Consider a graph $P_{a,b}$ with $a \ge 3$. Define $f : V(P_{a,4}) \to \{1, 2, 3, \cdots, 4a+1\}$ as follows:

$$\begin{aligned} f(u) &= a+1, \\ f(v) &= 4a+1, \end{aligned}$$

$$f(v_{a-j}^{(1)}) = \begin{cases} j, & 1 \le j \le \left\lceil \sqrt{4a+1} \right\rceil - 2, \\ j+1, & \left\lceil \sqrt{4a+1} \right\rceil - 1 \le j \le a-1, \end{cases}$$

$$f(v_j^{(2)}) = \begin{cases} a+3j-1, & 1 \le j \le a-1 \text{ and } j \text{ is odd}, \\ a+3j+1, & 1 \le j \le a-1 \text{ and } j \text{ is even}, \end{cases}$$

$$f(v_j^{(3)}) = \begin{cases} a+1+3j, & 1 \le j \le a-1 \text{ and } j \text{ is odd}, \\ a+3+3j, & 1 \le j \le a-1 \text{ and } j \text{ is even} \end{cases}$$

$$f(v_j^{(4)}) = \begin{cases} a+3+3j, & 1 \le j \le a-1 \text{ and } j \text{ is odd}, \\ a-1+3j, & 1 \le j \le a-1 \text{ and } j \text{ is even}. \end{cases}$$

Then, the induced edge labeling is obtained as follows:

$$f^*(v_{a-j}^{(1)}v_{a-j-1}^{(1)}) = \begin{cases} j+1, & 1 \le j \le \left\lceil \sqrt{4a+1} \right\rceil - 2, \\ j+2, & \left\lceil \sqrt{4a+1} \right\rceil - 1 \le j \le a-2, \end{cases}$$

$$\begin{aligned} f^*(v_{a-1}^{(1)}v) &= \left\lceil \sqrt{4a+1} \right\rceil, \\ f^*(uv_1^{(1)}) &= a+1, \\ f^*(uv_1^{(i)}) &= a+i, \text{ for } 2 \le i \le 4 \\ f^*(v_{a-1}^{(i)}v) &= 4a-3+i, \text{ for } 2 \le i \le 4 \text{ and} \end{aligned}$$

$$f^*(v_j^{(i)}v_{j+1}^{(i)}) = \begin{cases} a+2+3j, & i = 2 \text{ and } 1 \le j \le a-2, \\ a+7-i+3j & 3 \le i \le 4 \text{ and } 1 \le j \le a-2. \end{cases}$$

For $a = 2$, a C-Geometric mean labeling of $P_{2,4}$ is as shown in Figure 5.

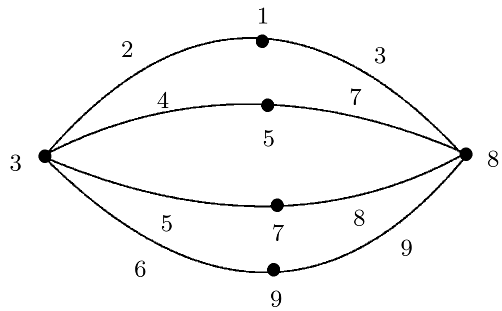

Figure 5 A C-Geometric mean labeling of $P_{2,4}$

Hence, the graph $P_{a,b}$ for $b \leq 4$ admits a C-Geometric mean labeling. Thus the graph $P_{a,b}$ for $b \leq 4$ is a C-Geometric mean graph. □

Theorem 2.10 *A graph P_a^b is a C-Geometric mean graph if $b \leq 3$.*

Proof Let $y_i, x_{ij1}, x_{ij2}, \cdots, x_{iji}, y_{i+1}$ be the vertices of the j^{th} path of i^{th} block of P_a^b, where $1 \leq i \leq a-1$ and $1 \leq j \leq b$. Obviously,

$$V(P_a^b) = \{y_i; 1 \leq i \leq a\} \bigcup \left(\bigcup_{i=1}^{a-1} \bigcup_{j=1}^{b} \{x_{ijk}; 1 \leq k \leq i\} \right)$$

$$E(P_a^b) = \bigcup_{i=1}^{a-1} \{y_i x_{ij1} : 1 \leq j \leq b\} \bigcup \left(\bigcup_{i=1}^{a-1} \bigcup_{j=1}^{b} \{x_{ijk} x_{ij(k+1)}; 1 \leq k \leq i-1\} \right)$$
$$\bigcup \left(\bigcup_{i=1}^{a-1} \{x_{iji} y_{i+1}; 1 \leq j \leq b\} \right)$$

Hence, $|V(P_a^b)| = \frac{ab(a-1)}{2} + a$ and $|E(P_a^b)| = \frac{b(a-1)(a+2)}{2}$.

Case 1. $b = 2$.

Notice that the graph P_a^2 is $G^*(p_1, p_2, \cdots, p_n)$. Applying Corollary 2.9, P_a^2 is a C-Geometric mean graph for $p_1 \neq 3$.

Case 2. $b = 3$.

Define $f : V(P_a^3) \to \left\{1, 2, 3, \cdots, \frac{3(a-1)(a+2)}{2} + 1\right\}$ as follows:

$$f(y_1) = 3,$$
$$f(y_i) = \frac{3(i-1)(i+2)}{2} + 1, \text{ for } 2 \leq i \leq a,$$
$$f(x_{111}) = 1,$$
$$f(x_{1j1}) = j + 3, \text{ for } 2 \leq j \leq 3,$$
$$f(x_{21k}) = 4k + 5, \text{ for } 1 \leq k \leq 2,$$
$$f(x_{22k}) = 5k + 5, \text{ for } 1 \leq k \leq 2,$$
$$f(x_{23k}) = 13 - k, \text{ for } 1 \leq k \leq 2$$

and for $3 \leq i \leq a-1$,

$$f(x_{ij1}) = \begin{cases} \frac{3(i-1)(i+2)}{2} + 2 + j, & 1 \leq j \leq 2, \\ \frac{3(i-1)(i+2)}{2} + 2j, & j = 3, \end{cases}$$

$$f(x_{ijk}) = \begin{cases} \frac{3(i-1)(i+2)}{2} + 2j + 3k - 1 & 1 \leq j \leq 2, 2 \leq k \leq i-1 \text{ and } k \text{ is even}, \\ \frac{3(i-1)(i+2)}{2} + 3k - 1, & j = 3, 2 \leq k \leq i-1 \text{ and } k \text{ is even}, \\ \frac{3(i-1)(i+2)}{2} + 2j + 3k - 3, & 1 \leq j \leq 3, 2 \leq k \leq i-1 \text{ and } k \text{ is odd}, \\ \frac{3(i-1)(i+2)}{2} + 3k - 1, & j = 1, k = i \text{ and } k \text{ is odd}, \\ \frac{3(i-1)(i+2)}{2} + 3k + j, & 2 \leq j \leq 3, k = i \text{ and } k \text{ is odd}, \\ \frac{3(i-1)(i+2)}{2} + 3k + j + 1, & 1 \leq j \leq 2, k = i \text{ and } k \text{ is even}, \\ \frac{3(i-1)(i+2)}{2} + 3k - 1, & j = 3, k = i \text{ and } k \text{ is even}. \end{cases}$$

Then, the induced edge labeling is as follows:

$$f^*(y_i x_{ij1}) = \frac{3(i-1)(i+2)}{2} + j + 1, \text{ for } 1 \leq j \leq 3 \text{ and } 2 \leq i \leq a-1,$$

$$f^*(y_1 x_{1j1}) = \begin{cases} 2, & j = 1, \\ j + 2, & 2 \leq j \leq 3, \end{cases}$$

$$f^*(x_{1j1} y_2) = \begin{cases} 3, & j = 1, \\ j + 4, & 2 \leq j \leq 3, \end{cases}$$

$$f^*(x_{2j1} x_{2j2}) = \begin{cases} 2j + 9, & 1 \leq j \leq 2, \\ 12, & j = 3, \end{cases}$$

$$f^*(x_{2j2} y_3) = \begin{cases} j + 14, & 1 \leq j \leq 2, \\ 14, & j = 3 \end{cases}$$

and for $3 \leq i \leq a-1$,

$$f^*(x_{ijk} x_{ijk+1}) = \begin{cases} \frac{3(i-1)(i+2)}{2} + 3k + 2j - 1, & 1 \leq k \leq i-1, \\ & \text{and } 1 \leq j \leq 2, \\ \frac{3(i-1)(i+2)}{2} + 3k + 2, & 1 \leq k \leq i-1, \\ & \text{and } j = 3, \end{cases}$$

$$\text{and } f^*(x_{iji} y_{i+1}) = \begin{cases} \frac{3i(i+3)}{2} + j - 2, & 1 \leq j \leq 3 \text{ and } i \text{ is odd}, \\ \frac{3i(i+3)}{2} + j - 1, & 1 \leq j \leq 2 \text{ and } i \text{ is even}, \\ \frac{3i(i+3)}{2} - 1, & j = 3 \text{ and } i \text{ is even}. \end{cases}$$

Hence, f is a C-Geometric mean labeling of P_a^b, for $b \leq 3$. Thus the graph P_a^b for $b \leq 3$ is a C-Geometric mean graph. □

Theorem 2.11 *Let G be a graph obtained from a path by identifying any of its edges by an edge of a cycle and none of the pendent edges is identified by an edge of a cycle of length 3. Then, G is a C-Geometric mean graph.*

Proof Let $v_1, v_2, \cdots, v_p$ be the vertices of the path on p vertices. Let m be the number of cycles are placed in a path in order to get G and the edges of the j^{th} cycle be identified with the edge (v_{i_j}, v_{i_j+1}) of the path having the length n_j and $n_1 \neq 3$ when $i_1 = 1$. For $1 \leq j \leq m$, the vertices of the j^{th} cycle be $v_{i_j,l}, 1 \leq l \leq n_j$ where $v_{i_j,1} = v_{i_j}$ and $v_{i_j,n_j} = v_{i_j+1}$.

Define $f : V(G) \rightarrow \left\{1, 2, 3, \cdots, \sum\limits_{j=1}^{m} n_j + p - m\right\}$ as follows:

$$\begin{aligned}
f(v_k) &= k, \text{ for } 1 \leq k \leq i_1,\\
f(v_{i_j}) &= i_j + \sum_{k=1}^{j-1}(n_k - 2) + j - 1, \text{ for } 1 \leq j \leq m,\\
f(v_{i_j+1}) &= f(v_{i_j}) + n_j, \text{ for } 1 \leq j \leq m,\\
f(v_{i_j+k}) &= f(v_{i_j+1}) + k - 1, \text{ for } 2 \leq k \leq i_{j+1} - i_j - 1 \text{ and } 1 \leq j \leq m-1,\\
f(v_{i_m+k+1}) &= f(v_{i_m+k}) + k - 1, \text{ for } 2 \leq k \leq p - i_m
\end{aligned}$$

and for $1 \leq j \leq m$,

$$f(v_{i_j,l}) = \begin{cases} f(v_{i_j}) + l - 1, & 2 \leq l \leq \left\lceil \sqrt{f(v_{i_j})f(v_{i_j+1})} \right\rceil - f(v_{i_j}) - 1,\\ f(v_{i_j}) + l, & \left\lceil \sqrt{f(v_{i_j})f(v_{i_j+1})} \right\rceil - f(v_{i_j}) \leq l \leq n_j - 1. \end{cases}$$

Then, the induced edge labeling is obtained as follows:

$$\begin{aligned}
f^*(v_k v_{k+1}) &= k + 1, \text{ for } 1 \leq k \leq i_1 - 1,\\
f^*(v_{i_j+k} v_{i_j+k+1}) &= v_{i_j+k} + 1, \text{ for } 1 \leq k \leq i_{j+1} - i_j - 1 \text{ and } 1 \leq j \leq m-1,\\
f^*(v_{i_m+k} v_{i_m+k+1}) &= f(v_{i_m+k}) + 1, \text{ for } 1 \leq k \leq p - i_m - 1
\end{aligned}$$

and for $1 \leq j \leq m$,

$$\begin{aligned}
f^*(v_{i_j,l} v_{i_j,l+1}) &= \begin{cases} f(v_{i_j}) + l, & 1 \leq l \leq \left\lceil \sqrt{f(v_{i_j})f(v_{i_j+1})} \right\rceil - f(v_{i_j}) - 1,\\ f(v_{i_j}) + l + 1, & \left\lceil \sqrt{f(v_{i_j})f(v_{i_j+1})} \right\rceil - f(v_{i_j}) \leq l \leq n_j - 1, \end{cases}\\
f^*(v_{i_j} v_{i_{j+1}}) &= \left\lceil \sqrt{f(v_{i_j})f(v_{i_{j+1}})} \right\rceil, \text{ for } 1 \leq j \leq m.
\end{aligned}$$

Hence, the graph G admits a C-Geometric mean labeling. Thus the graph G is obtained from a path by identifying any of its edges by an edge of a cycle and none of the pendent edges is identified by an edge of a cycle of length 3, is a C-Geometric mean graph. □

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Neighbourhood V_4-Magic Labeling of Some Shadow Graphs

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Abstract: The Klein 4-group,denoted by V_4 is an abelian group of order 4. It has elements $V_4 = \{0, a, b, c\}$ with $a + a = b + b = c + c = 0$ and $a + b = c, b + c = a, c + a = b$. A graph $G(V(G), E(G)$ is said to be neighbourhood V_4-magic if there exists a labeling $f : V(G) \rightarrow V_4\backslash\{0\}$ such that the induced mapping $N_f^+ : V(G) \rightarrow V_4$ defined by $N_f^+(v) = \sum_{u\in N(v)} f(u)$ is a constant map. If this constant is $p(p \neq 0)$,we say that f is a $p-$neighbourhood V_4-magic labeling of G and G a $p-$neighbourhood V_4-magic graph. If this constant is zero, we say that f is a $0-$neighghbourhood V_4-magic labeling of G and G a $0-$neighbourhood V_4-magic graph. In this paper, we discuss neighbourhood V_4-magic labeling of some shadow graphs.

Key Words: Klein-4-group, shadow graphs, a-neighbourhood V_4-magic graphs, 0-neighbourhood V_4-magic graphs, Smarandachely V_4-magic.

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§1. Introduction

Throughout this paper we consider simple, finite, connected and undirected graphs. For standard terminology and notation we follow [1] and [2]. For a detailed survey on graph labeling we refer [6]. The V_4-magic graphs were introduced by S. M. Lee et al. in 2002 [3]. We say that, a graph $G = (V(G), E(G))$, with vertex set $V(G)$ and edge set $E(G)$ is neighbourhood V_4-magic if there exists a labeling $f : V(G) \rightarrow V_4\backslash\{0\}$ such that the induced mapping $N_f^+ : V(G) \rightarrow V_4$ defined by $N_f^+(v) = \sum_{u\in N(v)} f(u)$ is a constant map. Otherwise, it is said to be *Smarandachely* V_4*-magic*, i.e., $\left|\left\{N_f^+(v), v \in V(G)\right\}\right| \geq 2$. If this constant is p, where p is any non zero element in V_4,then we say that f is a $p-$neighbourhood V_4-magic labeling of G and G is said to be a $p-$neighbourhood V_4-magic graph. If this constant is 0,then we say that f is a $0-$neighbourhood V_4-magic labeling of G and G is said to be a $0-$neighbourhood V_4-magic graph. We divide the class of neighbourhood V_4-magic graphs into the following three categories:

(1) $\Omega_a :=$ the class of all $a-$neighbourhood V_4-magic graphs;
(2) $\Omega_0 :=$ the class of all $0-$neighbourhood V_4-magic graphs, and
(3) $\Omega_{a,0} := \Omega_a \cap \Omega_0$.

The shadow graph $Sh(G)$ of a connected graph G is constructed by taking two copies of G say G_1 and G_2, join each vertex u in G_1; to the neighbours of the corresponding vertex v in G_2.The Bistar

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$B_{m,n}$is the graph obtained by joining the central vertex $K_{1,m}$ and $K_{1,n}$ by an edge [6]. The wheel graph W_n is defined as $W_n \simeq C_n + K_1$, where C_n for $n \geq 3$ is a cycle of length n.The Helm H_n is a graph obtained from the wheel graph W_n by attaching a pendant edge at each vertex of the cycle C_n [7].The Sunflower SF_n is obtained from a wheel with the central vertex w_0 and cycle $C_n = w_1w_2w_3\cdots w_nw_1$ and additional vertices $v_1, v_2, v_3, \cdots, v_n$ where v_i is joined by edges to w_i and w_{i+1} where $i+1$ is taken over modulo n [8].Jelly fish graph$J(m,n)$is obtained from a $4-$cycle $w_1w_2w_3w_4w_1$ by joining w_1 and w_3 with an edge and appending the central vertex of $K_{1,m}$ to w_2 and appending the central vertex of $K_{1,n}$ to w_4 [6]. The graph $P_2\Box P_n$ is called Ladder, it is denoted by L_n [5].The graph with vertex set$\{u_i, v_i : 0 \leq i \leq n+1\}$and edge set $\{u_iu_{i+1}, v_iv_{i+1} : 0 \leq i \leq n\} \cup \{u_iv_i : 1 \leq i \leq n\}$ is called the ladder L_{n+2}.The corona $P_n \odot K_1$ is called the comb graph CB_n.The Book graph B_n is the graph $S_n\Box P_2$, where S_n is the star with $n+1$ vertices and P_2 is the path on 2 vertices [5]. A gear graph G_n is obtained from the wheel graph by adding a vertex between every pair of adjacent vertices of the cycle. G_n has $2n+1$vertices and $3n$ edges [9]. This paper investigate neighbourhood V_4-magic labeling of shadow graphs of the above said graphs.

§2. Main Results

Theorem 2.1 *The graph $Sh(C_n) \in \Omega_a$ if and only if $n \equiv 0(mod\ 4)$.*

Proof Considering the shadow graph $Sh(C_n)$, let $\{u_1, u_2, u_3, \cdots, u_n\}$be the vertex set of first copy of C_n and let $\{v_1, v_2, v_3, \cdots, v_n\}$be the corresponding vertex set of second copy of C_n in order. Assume that $n \not\equiv 0(mod\ 4)$.Then either $n \equiv 1(mod\ 4)$ or $n \equiv 2(mod\ 4)$ or $n \equiv 3(mod\ 4)$. We show that in each these cases $Sh(C_n) \notin \Omega_a$.

Case 1. $n \equiv 1(mod\ 4)$

In this case $n = 4k+1$ for some $k \in \mathbb{N}$. Then $V(Sh(C_n)) = \{u_i, v_i : 1 \leq i \leq 4k+1\}$. If possible, let $Sh(C_n) \in \Omega_a$ with a labeling f. Then $N_f^+(u_2) = a$ implies that $f(u_1) + f(v_1) + f(u_3) + f(v_3) = a$, $N_f^+(u_4) = a$ implies that $f(u_3) + f(v_3) + f(u_5) + f(v_5) = a$. Proceeding like this, $N_f^+(u_{4k}) = a$ implies that $f(u_{4k-1}) + f(v_{4k-1}) + f(u_{4k+1}) + f(v_{4k+1}) = a$. Now consider $f(u_1) + f(v_1)$,then either $f(u_1) + f(v_1) = 0$ or $f(u_1) + f(v_1) = a$ or $f(u_1) + f(v_1) = b$ or $f(u_1) + f(v_1) = c$.

Subcase 1.1 $f(u_1) + f(v_1) = 0$

If $f(u_1) + f(v_1) = 0$,then $f(u_3) + f(v_3) = a, f(u_5) + f(v_5) = 0, f(u_7) + f(v_7) = a$, which implies that $f(u_{4k+1}) + f(v_{4k+1}) = 0$. Now $N_f^+(u_1) = a$ implies that $f(u_2) + f(v_2) = a, f(u_4) + f(v_4) = 0, f(u_6) + f(v_6) = a$. Proceeding like this we get $f(u_{4k}) + f(v_{4k}) = 0$. Therefore,$N_f^+(u_{4k+1}) = f(u_1) + f(v_1) + f(u_{4k}) + f(v_{4k}) = 0 + 0 = 0$, a contradiction.

Subcase 1.2 $f(u_1) + f(v_1) = a$

If $f(u_1) + f(v_1) = a$, then proceeding as in Subcase 1.1 we get $N_f^+(u_{4k+1}) = f(u_1) + f(v_1) + f(u_{4k}) + f(v_{4k}) = a + a = 0$, a contradiction.

Subcase 1.3 $f(u_1) + f(v_1) = b$

If $f(u_1)+f(v_1) = b$,then $f(u_3)+f(v_3) = c, f(u_5)+f(v_5) = b, f(u_7)+f(v_7) = c$, which implies that $f(u_{4k+1})+f(v_{4k+1}) = b$. Now,$N_f^+(u_1) = a$ gives $f(u_2)+f(v_2) = c, f(u_4)+f(v_4) = b, f(u_{4k})+f(v_{4k}) = b$. Therefore,$N_f^+(u_{4k+1}) = f(u_1) + f(v_1) + f(u_{4k}) + f(v_{4k}) = b + b = 0$, which is a contradiction.

Subcase 1.4 $f(u_1) + f(v_1) = c$

If $f(u_1)+f(v_1)=c$, then proceeding as in Subcase 1.3 we get $N_f^+(u_{4k+1})=f(u_1)+f(v_1)+f(u_{4k})+f(v_{4k})=c+c=0$, a contradiction.

Thus if $n\equiv 1(mod\ 4)$, we have $Sh(C_n)\notin\Omega_a$.

Case 2. $n\equiv 2(mod\ 4)$

In this case $n=4k+2$ for some $k\in\mathbb{N}$. Then $V(Sh(C_n))=\{u_i,v_i : 1\leq i\leq 4k+2\}$. If possible let $Sh(C_n)\in\Omega_a$ with a labeling f. Considering $f(u_1)+f(v_1)$, then either $f(u_1)+f(v_1)=0$ or $f(u_1)+f(v_1)=a$ or $f(u_1)+f(v_1)=b$ or $f(u_1)+f(v_1)=c$.

Subcase 2.1 $f(u_1)+f(v_1)=0$

If $f(u_1)+f(v_1)=0$,then $N_f^+(u_2)=a, f(u_3)+f(v_3)=a, f(u_5)+f(v_5)=0$, which implies that $f(u_{4k+1})+f(v_{4k+1})=0$. Therefore, $N_f^+(u_{4k+2})=f(u_1)+f(v_1)+f(u_{4k+1})+f(v_{4k+1})=0+0=0$, a contradiction.

Subcase 2.2 $f(u_1)+f(v_1)=a$

If $f(u_1)+f(v_1)=a$, then proceeding as in Subcase 2.1 we get $N_f^+(u_{4k+2})=f(u_1)+f(v_1)+f(u_{4k+1})+f(v_{4k+1})=a+a=0$, which is a contradiction.

Subcase 2.3 $f(u_1)+f(v_1)=b$

If $f(u_1)+f(v_1)=b$,then $N_f^+(u_2)=a$ implies that $f(u_3)+f(v_3)=c, f(u_5)+f(v_5)=b$, implies that $f(u_{4k+1})+f(v_{4k+1})=b$. Therefore, $N_f^+(u_{4k+2})=f(u_1)+f(v_1)+f(u_{4k+1})+f(v_{4k+1})=b+b=0$, which is a contradiction.

Subcase 2.4 $f(u_1)+f(v_1)=c$

If $f(u_1)+f(v_1)=c$, then proceeding as in Subcase 2.3 we get $N_f^+(u_{4k+2})=f(u_1)+f(v_1)+f(u_{4k+1})+f(v_{4k+1})=c+c=0$, a contradiction.

Thus if $n\equiv 2(mod\ 4)$, $Sh(C_n)\notin\Omega_a$.

Case 3. $n\equiv 3(mod\ 4)$

In this case $n=4k+3$ for some $k\in\mathbb{N}$. Then $V(Sh(C_n))=\{u_i,v_i : 1\leq i\leq 4k+3\}$. If possible let $Sh(C_n)\in\Omega_a$ with a labeling f. Considering $f(u_1)+f(v_1)$, then either $f(u_1)+f(v_1)=0$ or $f(u_1)+f(v_1)=a$ or $f(u_1)+f(v_1)=b$ or $f(u_1)+f(v_1)=c$.

Subcase 3.1 $f(u_1)+f(v_1)=0$

If $f(u_1)+f(v_1)=0$, then $N_f^+(u_2)=a$ gives $f(u_3)+f(v_3)=a$, $f(u_5)+f(v_5)=0$, $f(u_{4k+1})+f(v_{4k+1})=0$, $f(u_{4k+3})+f(v_{4k+3})=a$. Now, $N_f^+(u_1)=a$ implies that $f(u_2)+f(v_2)=0$, $f(u_4)+f(v_4)=a$, $f(u_{4k+2})+f(v_{4k+2})=0$. Therefore $N_f^+(u_{4k+3})=f(u_1)+f(v_1)+f(u_{4k+2})+f(v_{4k+2})=0+0=0$, which is a contradiction.

Subcase 3.2 $f(u_1)+f(v_1)=a$

If $f(u_1)+f(v_1)=a$, then proceeding as in Subcase 3.1 we get $N_f^+(u_{4k+3})=f(u_1)+f(v_1)+f(u_{4k+2})+f(v_{4k+2})=a+a=0$, a contradiction.

Subcase 3.3 $f(u_1)+f(v_1)=b$

If $f(u_1)+f(v_1)=b$, then $N_f^+(u_2)=a$ implies that $f(u_3)+f(v_3)=c, f(u_5)+f(v_5)=b, f(u_{4k+1})+f(v_{4k+1})=b, f(u_{4k+3})+f(v_{4k+3})=c$. Now, $N_f^+(u_1)=a$ implies that $f(u_2)+f(v_2)=b, f(u_4)+f(v_4)=$

$c, f(u_{4k+2}) + f(v_{4k+2}) = b$. Therefore, $N_f^+(u_{4k+3}) = f(u_1) + f(v_1) + f(u_{4k+2}) + f(v_{4k+2}) = b + b = 0$, which is a contradiction.

Subcase 3.4 $f(u_1) + f(v_1) = c$

If $f(u_1) + f(v_1) = c$, then proceeding as in Subcase 3.3 we get $N_f^+(u_{4k+3}) = f(u_1) + f(v_1) + f(u_{4k+2}) + f(v_{4k+2}) = c + c = 0$, a contradiction.

Thus if $n \equiv 3(mod\ 4)$, we also have $Sh(C_n) \notin \Omega_a$. Therefore, $n \not\equiv 0(mod\ 4)$ implies that $Sh(C_n) \notin \Omega_a$.

Conversely if $n \equiv 0(mod\ 4)$, We define $f : V(Sh(C_n)) \to V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 2(mod\ 4), \\ c & \text{if} \quad i \equiv 0, 3(mod\ 4) \end{cases} \qquad \text{and} \quad f(v_i) = a \quad \text{for} \quad 1 \le i \le n.$$

Then, f is an $a-$neighbourhood V_4-magic labeling for $Sh(C_n)$. This completes the proof of the theorem. □

Theorem 2.2 *$Sh(C_n) \in \Omega_0$ for all $n \ge 3$.*

Proof The degree of each vertex in $Sh(C_n)$is 4. By labeling all the vertices by a, we get $N_f^+(u) = 0$ for all $u \in V(Sh(C_n))$. □

Corollary 2.3 *$Sh(C_n) \in \Omega_{a,0}$ if and only if $n \equiv 0(mod\ 4)$.*

Proof The proof is obviously follows from Theorems 2.1 and 2.2. □

Theorem 2.4 *The graph $Sh(P_n) \in \Omega_0$ for all $n \ge 2$.*

Proof If we label all the vertices by a,we get $G \in \Omega_0$. □

Theorem 2.5 *$Sh(P_n) \in \Omega_a$ for $n \equiv 0, 2, 3(mod\ 4)$.*

Proof Let G be the shadow graph $Sh(P_n)$, and let $\{u_i : 1 \le i \le n\}$and$\{v_i : 1 \le i \le n\}$ be the vertex sets of first and second copy of P_n respectively.

Case 1. $n \equiv 0(mod\ 4)$

Define $f : V(G) \to V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} a & \text{if} \quad i \equiv 0, 1(mod\ 4), \\ b & \text{if} \quad i \equiv 2, 3(mod\ 4), \end{cases}$$

$$f(v_i) = \begin{cases} a & \text{if} \quad i \equiv 0, 1(mod\ 4), \\ c & \text{if} \quad i \equiv 2, 3(mod\ 4). \end{cases}$$

Case 2. $n \equiv 2(mod\ 4)$

Define $f : V(G) \to V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} a & \text{if} \quad i \equiv 0, 3(mod\ 4), \\ b & \text{if} \quad i \equiv 1, 2(mod\ 4), \end{cases}$$

$$f(v_i) = \begin{cases} a & \text{if} \quad i \equiv 0, 3(mod\ 4), \\ c & \text{if} \quad i \equiv 1, 2(mod\ 4). \end{cases}$$

Case 3. $n \equiv 3(mod\ 4)$

Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 2(mod\ 4), \\ a & \text{if} \quad i \equiv 0, 3(mod\ 4), \end{cases}$$

$$f(v_i) = \begin{cases} c & \text{if} \quad i \equiv 1, 2(mod\ 4), \\ a & \text{if} \quad i \equiv 0, 3(mod\ 4). \end{cases}$$

In all the above cases, we have $N_f^+(u_i) = N_f^+(v_i) = a$ for $1 \le i \le n$. Therefore, $Sh(P_n) \in \Omega_a$ for $n \equiv 0, 2, 3(mod\ 4)$. □

Theorem 2.6 *$Sh(P_n) \notin \Omega_a$ for $n \equiv 1(mod\ 4)$.*

Proof Consider the shadow graph $Sh(P_n)$ with $n \equiv 1(mod\ 4)$. Let $\{u_i : 1 \le i \le 4k+1\}$ and$\{v_i : 1 \le i \le 4k+1\}$ be the vertex sets of first and second copy of P_n respectively. Assume that $Sh(P_n) \in \Omega_a$ with a labeling f. Since $N_f^+(u_1) = a$, we have either $f(u_2) = b$ and $f(v_2) = c$ or $f(u_2) = c$ and $f(v_2) = b$. Without loss of generality assume that $f(u_2) = b$ and $f(v_2) = c$. Then $f(u_{4k}) = f(v_{4k})$ implies that $N_f^+(u_{4k+1}) = 0$, a contradiction. Therefore, $Sh(P_n) \notin \Omega_a$. □

Corollary 2.7 *$Sh(P_n) \in \Omega_{a,0}$ for $n \equiv 0, 2, 3(mod\ 4)$.*

Proof The proof directly follows from theorems 2.4 and 2.5. □

Theorem 2.8 *$Sh(K_{1,n}) \in \Omega_a$ for all $n \in \mathbb{N}$.*

Proof Let $V = \{u_i, v_i : 0 \le i \le n\}$ be the vertex set of $Sh(K_{1,n})$ where $\{u_i : 0 \le i \le n\}$ and $\{v_i : 0 \le i \le n\}$ are the vertex sets of first and second copy of $K_{1,n}$ with apex u_0, v_0 respectively. Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 0, 1, \\ a & \text{if} \quad i = 2, 3, \cdots, n, \end{cases}$$

$$f(v_i) = \begin{cases} c & \text{if} \quad i = 0, 1, \\ a & \text{if} \quad i = 2, 3, \cdots, n. \end{cases}$$

Then, $N_f^+(u_i) = N_f^+(v_i) = a$ for all $0 \le i \le n$. This completes the proof. □

Theorem 2.9 *$Sh(K_{1,n}) \in \Omega_0$ for all $n \in \mathbb{N}$.*

Proof If we label all the vertices by a,we get $Sh(K_{1,n}) \in \Omega_0$. □

Corollary 2.10 *$Sh(K_{1,n}) \in \Omega_{a,0}$ for all $n \in \mathbb{N}$.*

Proof The proof obviously follows from Theorems 2.8 and 2.9. □

Theorem 2.11 *$Sh(B_{m,n}) \in \Omega_0$ for all m and n.*

Proof Labeling all the vertices by a, we get $Sh(B_{m,n}) \in \Omega_0$ for all m and n. □

Theorem 2.12 *$Sh(B_{m,n}) \in \Omega_a$ for all $m > 1$ and $n > 1$.*

Proof Let $V_1 = \{u, v, u_1, u_2, \cdots, u_m, v_1, v_2, v_3, \cdots, v_n\}$ be the vertex set of first copy of $B_{m,n}$ and $V_2 = \{u', v', u'_1, u'_2, \cdots, u'_m, v'_1, v'_2, v'_3, \cdots, v'_n\}$ be the corresponding vertex set of second copy of $B_{m,n}$, where u_i, v_i are pendant vertices adjacent to u, v respectively. Then $V(Sh(B_{m,n})) = V_1 \cup V_2$.

Define $f : V(Sh(B_{m,n})) \rightarrow V_4\backslash\{0\}$ as:

$f(u) = f(v) = b$;

$f(u') = f(v') = c$;

$f(u_i) = f(u'_i) = a$ for $1 \leq i \leq m$;

$f(v_i) = f(v'_i) = a$ for $1 \leq i \leq n$.

Then, f is an $a-$neighbourhod labeling of $Sh(B_{m,n})$. This completes the proof. □

Corollary 2.13 *$Sh(B_{m,n}) \in \Omega_{a,0}$ for all $m > 1$ and $n > 1$.*

Proof The proof follows from Theorems 2.11 and 2.12. □

Theorem 2.14 *$Sh(W_n) \in \Omega_0$ for all $n \geq 3$.*

Proof The degree of a vertex in $Sh(W_n)$ is either 6 or $2n$. If we label all the vertices by a, we get $N_f^+(u) = 0$ for all $u \in V(Sh(W_n))$. □

Theorem 2.15 *$Sh(W_n) \in \Omega_a$ for all $n \equiv 1(mod\ 2)$.*

Proof Let $V_1 = \{u_0, u_1, u_2, \cdots, u_n\}$ be the vertex set of first copy of W_n with central vertex u_0 and let $V_2 = \{v_0, v_1, v_2, \cdots, v_n\}$ be the corresponding vertex set of second copy of $W_n$with central vertex v_0. Then, $V = V(Sh(W_n)) = V_1 \cup V_2$. Define $f : V \rightarrow V_4\backslash\{0\}$ as:

$$f(u_i) = b \quad \text{if} \quad i = 0, 1, 2, 3, \cdots, n,$$

$$f(v_i) = c \quad \text{if} \quad i = 0, 1, 2, 3, \cdots, n.$$

Then, $N_f^+(u_i) = N_f^+(v_i) = a$ for all $i = 0, 1, 2, \cdots, n$. □

Corollary 2.16 *$Sh(W_n) \in \Omega_{a,0}$ for all $n \equiv 1(mod\ 2)$.*

Proof The proof directly follows from Theorems 2.14 and 2.15. □

Theorem 2.17 *$Sh(W_n) \in \Omega_a$ for all $n \equiv 2(mod\ 4)$.*

Proof Let $V_1 = \{u_0, u_1, u_2, \cdots, u_n\}$ be the vertex set of first copy of W_n with central vertex u_0 and let $V_2 = \{v_0, v_1, v_2, \cdots, v_n\}$ be the vertex set of second copy with central vertex v_0.Then $V(Sh(W_n)) = V_1 \cup V_2$. Define $f : V(Sh(W_n)) \rightarrow V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} a & \text{if} \quad i \equiv 1, 3(mod\ 4), \\ c & \text{if} \quad i \equiv 0, 2(mod\ 4), \end{cases}$$

$$f(v_i)=\begin{cases} a & \text{if} \quad i\equiv 1,3(mod\ 4),\\ b & \text{if} \quad i\equiv 0,2(mod\ 4).\end{cases}$$

Clearly, $N_f^+(u_i)=N_f^+(v_i)=a$ for all $i=0,1,2,\ldots,n$. Hence $Sh(W_n)\in\Omega_a$. □

Corollary 2.18 *$Sh(W_n)\in\Omega_{a,0}$ for all $n\equiv 2(mod\ 4)$.*

Proof The proof directly follows from Theorems 2.14 and 2.17. □

Theorem 2.19 *$Sh(H_n)\in\Omega_0$ for all $n\geq 3$.*

Proof In $Sh(H_n)$, degree of vertices are either 2 or 8 or $2n$. If we label all the vertices by a, we get $N_f^+(u)=0$ for all $u\in V(Sh(H_n))$. □

Theorem 2.20 *$Sh(H_n)$ admits $a-$neighbourhood V_4-magic labeling for all $n\equiv 1(mod\ 2)$.*

Proof Consider the shadow graph $Sh(H_n)$. Let v be central vertex,$v_1,v_2,v_3,\cdots,v_n$ be the rim vertices and $u_1,u_2,u_3,\cdots,u_n$ be the pendant vertices adjacent to $v_1,v_2,v_3,\cdots,v_n$ in the first copy of H_n and let $v',v_1',v_2',v_3',\cdots,v_n',u_1',u_2',u_3',\cdots,u_n'$ be the corresponding vertices in the second copy of H_n. Then $V(Sh(H_n))=\{v,v',v_i,v_i',u_i,u_i':1\leq i\leq n\}$. We define $f:V(Sh(H_n))\to V_4\backslash\{0\}$ as:

$$f(v)=a \quad \text{and} \quad f(v_i)=f(u_i)=b \quad \text{for} \quad i=1,2,3,\ldots,n,$$
$$f(v')=a \quad \text{and} \quad f(v_i')=f(u_i')=c \quad \text{for} \quad i=1,2,3,\ldots,n.$$

Obviously, f is an $a-$neighbourhood V_4-magic labeling of $Sh(H_n)$. □

Corollary 2.21 *$Sh(H_n)\in\Omega_{a,0}$ for all $n\equiv 1(mod\ 2)$.*

Proof The proof directly follows from Theorems 2.19 and 2.20. □

Theorem 2.22 *$Sh(SF_n)$ admits $a-$neighbourhood V_4-magic labeling for all $n\equiv 2(mod\ 4)$.*

Proof Considering $Sh(SF_n)$, let the vertex set of first copy of SF_n be $V_1=\{w,w_i,v_i:1\leq i\leq n\}$ where w is the central vertex, $w_1,w_2,w_3,\cdots,w_n$ are vertices of the cycle and v_i is the vertex joined by edges to w_i and w_{i+1} where $i+1$ is taken over modulo n. Let $V_2=\{w',w_i',v_i':1\leq i\leq n\}$ be the corresponding vertex set of second copy of SF_n. Then $V(Sh(SF_n))=V_1\cup V_2$. Define $f:V(Sh(SF_n))\to V_4\backslash\{0\}$ as:

$$f(w_i)=\begin{cases} b & \text{if} \quad i\equiv 1(mod\ 2),\\ c & \text{if} \quad i\equiv 0(mod\ 2),\end{cases}$$

$$f(v_i)=\begin{cases} b & \text{if} \quad i\equiv 1(mod\ 2),\\ c & \text{if} \quad i\equiv 0(mod\ 2),\end{cases}$$

$$f(w)=f(w')=f(w_i')=f(v_i')=a \text{ for } i=1,2,3,\cdots,n.$$

Then f is an $a-$neighbourhood V_4-magic labeling of $Sh(SF_n)$. □

Theorem 2.23 *$Sh(SF_n)$ admits $0-$neighbourhood V_4-magic labeling for all n.*

Proof If we label all the vertices by a, we get $N_f^+(u)=0$ for all $u\in V(Sh(SF_n))$. □

Theorem 2.24 *$Sh(SF_n) \in \Omega_{a,0}$ for all $n \equiv 2(mod\ 4)$.*

Proof The proof is obviously follows from Theorems 2.22 and 2.23. □

Theorem 2.25 *$Sh(C_n \odot K_2) \in \Omega_a$ for all $n \equiv 0(mod\ 4)$.*

Proof Let G bet the shadow graph $Sh(C_n \odot K_2)$. Let $V_1 = \{u_i, v_i, w_i : 1 \leq i \leq n\}$ be the vertex set of first copy of $C_n \odot K_2$, where $u_i's$ are vertices of C_n and v_j, w_j are the vertices on j^{th} copy of K_2 and let $V_2 = \{u_i', v_i', w_i' : 1 \leq i \leq n\}$ be the corresponding vertex set of second copy of $C_n \odot K_2$. Then $V(G) = V_1 \cup V_2$. Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 2(mod\ 4), \\ c & \text{if} \quad i \equiv 0, 3(mod\ 4), \end{cases}$$

$$f(v_i) = \begin{cases} c & \text{if} \quad i \equiv 1, 2(mod\ 4), \\ b & \text{if} \quad i \equiv 0, 3(mod\ 4), \end{cases}$$

$$f(w_i) = \begin{cases} c & \text{if} \quad i \equiv 1, 2(mod\ 4), \\ b & \text{if} \quad i \equiv 0, 3(mod\ 4), \end{cases}$$

$$f(u_i') = f(v_i') = f(w_i') = a \text{ for } i = 1, 2, 3, \cdots, n.$$

Then f is an $a-$neighbourhood V_4-magic labeling of $Sh(C_n \odot K_2)$. □

Theorem 2.26 *$Sh(C_n \odot K_2) \in \Omega_0$ for all n.*

Proof By labeling all the vertices of $Sh(C_n \odot K_2)$ by a,we get $N_f^+(u) = 0$. □

Corollary 2.27 *$Sh(C_n \odot K_2) \in \Omega_{a,0}$ for all $n \equiv 0(mod\ 4)$.*

Proof The proof follows from Theorems 2.25 and 2.26. □

Theorem 2.28 *$Sh(C_n \odot \overline{K}_m) \in \Omega_a$ for all m and $n \geq 3$.*

Proof Let G be the shadow graph $Sh(C_n \odot \overline{K}_m)$. Let $u_1, u_2, u_3, \cdots, u_n$ be the rim vertices of first copy of $C_n \odot \overline{K}_m$ and $\{u_{i1}, u_{i2}, u_{i3}, \cdots, u_{im}\}$ be the set of pendant vertices adjacent to u_i for $1 \leq i \leq n$ in $C_n \odot \overline{K}_m$ and let $u_1', u_2', u_3', \cdots, u_n'$ be the rim vertices of second copy of $C_n \odot \overline{K}_m$ and $\{u_{i1}', u_{i2}', u_{i3}', \cdots, u_{im}'\}$ be the set of pendant vertices adjacent to u_i' for $1 \leq i \leq n$ in second copy of $C_n \odot \overline{K}_m$. Here we consider two cases.

Case 1. $m = 1$

Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(u_i) = f(u_{i,1}) = b \quad \text{for} \quad i = 1, 2, 3, \cdots, n.$$

$$f(u_i') = f(u_{i,1}') = c \quad \text{for} \quad i = 1, 2, 3, \cdots, n.$$

Case 2. $m \geq 2$

Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(u_i) = b \quad \text{for} \quad i = 1, 2, 3, \cdots, n.$$

$$f(u'_i) = c \quad \text{for} \quad i = 1,2,3,\cdots,n.$$
$$f(u'_{ij}) = a \quad \text{for} \quad i = 1,2,3,\cdots,n.$$
$$f(u_{ij}) = \begin{cases} b & \text{if} \quad j = 1, \\ c & \text{if} \quad j = 2, \\ a & \text{if} \quad j > 2. \end{cases}$$

Obviously, f is an $a-$neighbourhood V_4-magic labeling of $Sh(C_n \odot \overline{K}_m)$. □

Theorem 2.29 *$Sh(C_n \odot \overline{K}_m) \in \Omega_0$ for all m and $n \geq 3$.*

Proof Labeling all the vertices by a,we get $Sh(C_n \odot \overline{K}_m) \in \Omega_0$. □

Corollary 2.30 *$Sh(C_n \odot \overline{K}_m) \in \Omega_{a,0}$ for all m and $n \geq 3$.*

Proof The proof directly follows from Theorems 2.28 and 2.29. □

Theorem 2.31 *$Sh(J(m,n)) \in \Omega_0$ for all m and n.*

Proof Labeling all the vertices by a, we get $Sh(J(m,n)) \in \Omega_0$. □

Theorem 2.32 *$Sh(J(m,n)) \in \Omega_a$ for all m and n.*

Proof Let G be the graph $Sh(J(m,n))$. Let $V_1 = \{w_i, u_j, v_k : 1 \leq i \leq 4, 1 \leq j \leq m, 1 \leq k \leq n\}$ and $E_1 = \{w_1w_2, w_2w_3, w_3w_4, w_4w_1, w_1w_3\} \cup \{w_2u_j : 1 \leq j \leq m\} \cup \{w_4v_j : 1 \leq j \leq n\}$ be the vertex and edge set of first copy of $J(m,n)$ and let $V_2 = \{w'_i, u'_j, v'_k : 1 \leq i \leq 4, 1 \leq j \leq m, 1 \leq k \leq n\}$ be the corresponding vertex set of second copy of $J(m,n)$. Then $V(G) = V_1 \cup V_2$. Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(w_i) = b \quad \text{for} \quad i = 1,2,3,4;$$
$$f(w'_i) = c \quad \text{for} \quad i = 1,2,3,4;$$

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 1, \\ a & \text{if} \quad i \geq 2, \end{cases} \qquad f(u'_i) = \begin{cases} c & \text{if} \quad i = 1, \\ a & \text{if} \quad i \geq 2, \end{cases}$$

$$f(v_i) = \begin{cases} b & \text{if} \quad i = 1, \\ a & \text{if} \quad i \geq 2, \end{cases} \qquad f(v'_i) = \begin{cases} c & \text{if} \quad i = 1, \\ a & \text{if} \quad i \geq 2. \end{cases}$$

Then, f is an $a-$neighbourhood V_4-magic labeling of $Sh(J(m,n))$. □

Corollary 2.33 *$Sh(J(m,n)) \in \Omega_{a,0}$ for all m and n.*

Proof The proof directly follows from Theorems 2.31 and 2.32. □

Theorem 2.34 *$Sh(L_n) \in \Omega_0$ for all n.*

Proof By labeling all the vertices by a,we get $Sh(L_n) \in \Omega_0$ for all n. □

Theorem 2.35 *$Sh(L_n) \in \Omega_a$ for all $n \equiv 2(mod\ 3)$.*

Proof Consider $Sh(L_n)$ with $n \equiv 2(mod\ 3)$. Let $V_1 = \{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set of first copy of L_n with edge set $E_1 = \{u_i u_{i+1}, v_i v_{i+1}, u_j v_j : 1 \leq i \leq n-1, 1 \leq j \leq n\}$. Also let $V_2 = \{u_i', v_i' : 1 \leq i \leq n\}$ be the corresponding set of vertices in second copy of L_n. Then $V = V(Sh(L_n)) = V_1 \cup V_2$. Define $f : V \to V_4 \backslash \{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i \equiv 1, 2(mod\ 6), \\ c & \text{if} \quad i \equiv 4, 5(mod\ 6), \\ a & \text{if} \quad i \equiv 0, 3(mod\ 6), \end{cases}$$

$$f(v_i) = \begin{cases} c & \text{if} \quad i \equiv 1, 2(mod\ 6), \\ b & \text{if} \quad i \equiv 4, 5(mod\ 6), \\ a & \text{if} \quad i \equiv 0, 3(mod\ 6), \end{cases}$$

$$f(u_i') = a \quad \text{for} \quad i = 1, 2, 3, \cdots, n,$$

$$f(v_i') = a \quad \text{for} \quad i = 1, 2, 3, \cdots, n.$$

Then, f is an a-neighbourhood V_4-magic labeling of $Sh(L_n)$. □

Corollary 2.36 *$Sh(L_n) \in \Omega_{a,0}$ for all $n \equiv 2(mod\ 3)$.*

Proof The proof directly follows from Theorems 2.34 and 2.35. □

Theorem 2.37 *$Sh(L_{n+2}) \in \Omega_0$ for all $n \in \mathbb{N}$.*

Proof By labeling all the vertices by a,we get $Sh(L_{n+2}) \in \Omega_0$ for all n. □

Theorem 2.38 *$Sh(L_{n+2}) \in \Omega_a$ for all $n \in \mathbb{N}$.*

Proof Let G be the shadow graph $Sh(L_{n+2})$. Let $V_1 = \{u_i, v_i : 0 \leq i \leq n+1\}$ and $E_1 = \{u_i u_{i+1}, v_i v_{i+1} : 0 \leq i \leq n\} \cup \{u_i v_i : 1 \leq i \leq n\}$ be the vertex and edge set of first copy of L_{n+2} and let $V_2 = \{u_i', v_i' : 0 \leq i \leq n+1\}$ be the corresponding set of vertices in second copy of L_{n+2}. Define $f : V(Sh(L_{n+2})) \to V_4 \backslash \{0\}$ as:

$$f(u_i) = f(v_i) = b \quad \text{for} \quad i = 0, 1, 2, 3, \cdots, n+1,$$

$$f(u_i') = f(v_i') = c \quad \text{for} \quad i = 0, 1, 2, 3, \cdots, n+1,$$

Then, $N_f^+(u) = a$ for all vertices u in $Sh(L_{n+2})$. □

Corollary 2.39 *$Sh(L_{n+2}) \in \Omega_{a,0}$ for all $n \in \mathbb{N}$.*

Proof The proof directly follows from Theorems 2.37 and 2.38. □

Theorem 2.40 *$Sh(CB_n) \in \Omega_a$ for all $n > 1$.*

Proof Let $\{u_i, v_i : 1 \leq i \leq n\}$ be the vertex set of first copy of CB_n where v_i $(1 \leq i \leq n)$ are the pendant vertices adjacent to u_i $(1 \leq i \leq n)$. Let $\{u_i', v_i' : 1 \leq i \leq n\}$ be the corresponding set of vertices in second copy of CB_n. Define $f : V(Sh(CB_n)) \to V_4\backslash\{0\}$ as

$$\begin{aligned} f(u_i) &= b \quad \text{if} \quad 1 \leq i \leq n; \\ f(u_i') &= c \quad \text{if} \quad 1 \leq i \leq n; \\ f(v_i) &= \begin{cases} a & \text{if} \quad i = 1 \text{ or } n, \\ b & \text{if} \quad 1 < i < n, \end{cases} \\ f(v_i') &= \begin{cases} a & \text{if} \quad i = 1 \text{ or } n, \\ c & \text{if} \quad 1 < i < n. \end{cases} \end{aligned}$$

Then f is an $a-$neighbourhood V_4-magic labeling of CB_n. □

Theorem 2.41 *$Sh(CB_n) \in \Omega_0$ for all $n \in \mathbb{N}$.*

Proof By labeling all the vertices by a,we get $Sh(CB_n) \in \Omega_0$. □

Corollary 2.42 *$Sh(CB_n) \in \Omega_{a,0}$ for all $n > 1$.*

Proof The proof directly follows from Theorems 2.40 and 2.41. □

Theorem 2.43 *$Sh(K_{m,n}) \in \Omega_a$ for all $m > 1$ and $n > 1$.*

Proof Let G be the shadow graph $Sh(K_{m,n})$. Let $X = \{u_1, u_2, u_3, \cdots, u_m\}$ and $Y = \{v_1, v_2, v_3, \cdots, v_n\}$ be the bipartition of the first copy of $K_{m,n}$ and let $X' = \{u_1', u_2', u_3', \ldots, u_m'\}$ and $Y' = \{v_1', v_2', v_3', \ldots, v_n'\}$ be the corresponding bipartition second copy of $K_{m,n}$. Define $f : V(G) \to V_4\backslash\{0\}$ as:

$$f(u_i) = \begin{cases} b & \text{if} \quad i = 1, \\ c & \text{if} \quad i = 2, \\ a & \text{if} \quad i > 2, \end{cases} \qquad f(v_j) = \begin{cases} b & \text{if} \quad j = 1, \\ c & \text{if} \quad j = 2, \\ a & \text{if} \quad j > 2, \end{cases}$$

$$f(u_i') = a \ \text{ for } 1 \leq i \leq m \text{ and } f(v_j') = a \ \text{ for } 1 \leq j \leq n.$$

Then f is an $a-$neighbourhood V_4-magic labeling of $Sh(K_{m,n})$. This completes the proof of the theorem. □

Theorem 2.44 *$Sh(K_{m,n}) \in \Omega_0$ for all $m, n \in \mathbb{N}$.*

Proof Labeling all the vertices by a, we get $Sh(K_{m,n}) \in \Omega_0$. □

Corollary 2.45 *$Sh(K_{m,n}) \in \Omega_{a,0}$ for all $m > 1$ and $n > 1$.*

Proof The proof directly follows from Theorems 2.43 and 2.44. □

Theorem 2.46 *$Sh(B_n) \in \Omega_a$ for all $n \equiv 1(mod\ 2)$.*

Proof Let G be the shadow graph $Sh(B_n)$. Let vertex set of first copy of B_n be $V_1 = \{(u, v_j), (u_i, v_j) : 1 \leq i \leq n, 1 \leq j \leq 2\}$, where $\{u, u_1, u_2, u_3, \cdots, u_n\}$ and $\{v_1, v_2\}$ be the vertex sets of S_n and P_2 respectively, and u be the central vertex, $u_i's$ are pendant vertices in S_n. Also let $V_2 = \{(u', v_j'), (u_i', v_j') : 1 \leq i \leq n, 1 \leq j \leq 2\}$ be the corresponding vertex set of second copy of B_n. Then $V(G) = V_1 \cup V_2$.

Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(u, v_j) = \begin{cases} b & \text{if} \quad j = 1, \\ c & \text{if} \quad j = 2, \end{cases} \qquad \text{and} \qquad f(u_i, v_j) = \begin{cases} b & \text{if} \quad j = 1 \text{ and } 1 \le i \le n, \\ c & \text{if} \quad j = 2 \text{ and } 1 \le i \le n, \end{cases}$$

$$f(u', v'_j) = a \text{ for } j = 1, 2 \text{ and } f(u'_i, v'_j) = a \text{ for } 1 \le i \le n, \ 1 \le j \le 2.$$

Clearly, f is an $a-$neighbourhood V_4-magic labeling of $Sh(B_n)$. □

Theorem 2.47 *$Sh(B_n) \in \Omega_0$ for all $n \in \mathbb{N}$.*

Proof By labeling all the vertices by a, we get $Sh(B_n) \in \Omega_0$. □

Corollary 2.48 *$Sh(B_n) \in \Omega_{a,0}$ for all $n \equiv 1(mod\ 2)$.*

Proof The proof follows from Theorems 2.46 and 2.47. □

Theorem 2.49 *$Sh(G_n) \in \Omega_0$ for all n.*

Proof The degree of vertices in $Sh(B_n)$ is either 4 or 6 or $2n$. If we label all the vertices by a, we get $N_f^+(u) = 0$ for all $u \in V(Sh(G_n))$. □

Theorem 2.50 *$Sh(G_n) \in \Omega_a$ for all $n \equiv 2(mod\ 4)$.*

Proof Let G be the shadow graph $Sh(G_n)$. Let $V_1 = \{u, u_i : 1 \le i \le 2n\}$ and $E_1 = \{uu_{2i-1} : 1 \le i \le n\} \cup \{u_i u_{i+1} : 1 \le i \le 2n-1\} \cup \{u_{2n}u_1\}$ be the vertex and edge set of first copy of G_n. Let $V_2 = \{u', u'_i : 1 \le i \le 2n\}$ be the corresponding vertex set of second copy of G_n.Then $V(G) = V_1 \cup V_2$. Define $f : V(G) \to V_4 \backslash \{0\}$ as:

$$f(u) = b, \ f(u') = c \text{ and } f(u_i) = a \text{ for } 1 \le i \le 2n,$$

$$f(u'_i) = \begin{cases} a & \text{if} \quad i \equiv 0(mod\ 4), \\ b & \text{if} \quad i \equiv 1(mod\ 4), \\ a & \text{if} \quad i \equiv 2(mod\ 4), \\ c & \text{if} \quad i \equiv 3(mod\ 4). \end{cases}$$

Then f is an $a-$neighbourhood V_4-magic labeling for $Sh(G_n)$. □

Corollary 2.51 *$Sh(G_n) \in \Omega_{a,0}$ for all $n \equiv 2(mod\ 4)$.*

Proof The proof directly follows from Theorems 2.49 and 2.50. □

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Open Packing Number of Triangular Snakes

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Abstract: A set $S \subseteq V(G)$ of vertices in a graph G is called a packing of G if the closed neighborhood of the vertices of S are pairwise disjoint in G. A subset S of $V(G)$ is called an open packing of G if the open neighborhood of the vertices of S are pairwise disjoint in G. We have investigated exact value of these parameters for triangular snakes.

Key Words: Neighborhood, packing, Smarandache k-packing, open packing.

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§1. Introduction

We begin with the finite, connected and undirected graph $G = (V(G), E(G))$ without multiple edges and loops. For a vertex $v \in V(G)$, the open neighborhood $N(v)$ of v is defined as $N(v) = \{u \in V(G)/uv \in E(G)\}$ and the closed neighborhood $N[v] = \{v\} \cup N(v)$. We denote the degree of a vertex $v \in V(G)$ in a graph G by $d_G(v)$. The minimum degree among the vertices of G is denoted by $\delta(G)$ and the maximum degree among the vertices of G is denoted by $\Delta(G)$. For any real number n, $\lfloor n \rfloor$ denotes the greatest integer not greater than that n and $\lceil n \rceil$ denotes the smallest integer not less than that n. For the various graph theoretic notations and terminology, we follows West [8] and Haynes et al. [3].

Definition 1.1 *The triangular snake T_n is obtained from the path P_n by replacing every edge of a path by a triangle C_3.*

Definition 1.2 *An alternate triangular snake AT_n is obtained from a path P_n with vertices $u_1, u_2, \cdots, u_n$ by joining u_i and u_{i+1} (alternately) to a new vertex v_i. That is every alternate edge of a path is replaced by C_3.*

Definition 1.3 *The double triangular snake $D(T_n)$ is obtained from a path P_n with vertices $v_1, v_2, \cdots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, \cdots, n-1$ and to a new vertex u_i for $i = 1, 2, \cdots, n-1$.*

Definition 1.4 *A double alternate triangular snake $D(AT_n)$ consists of two alternate triangular snakes which have a common path.*

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A packing of a graph G is a set of vertices whose closed neighborhoods are pairwise disjoint. Generally, a Smarandache k-packing of a graph G is a set of vertices whose closed neighborhoods intersect just in k vertices, and disjoint if $k = 0$. Equivalently, a packing of a graph G is a set of vertices whose elements are pairwise at distance at least 3 apart in G. The maximum cardinality of a packing set of G is called the packing number and it is denoted by $\rho(G)$. This concept was introduced by Biggs [1].

A subset S of $V(G)$ is an open packing of G if the open neighborhoods of the vertices of S are pairwise disjoint in G. The maximum cardinality of an open packing set is called the open packing number and is denoted by ρ^o. This concept was introduced by Henning and Slater [5]. A brief account of on open packing and its related concepts can be found in [2,4,6,7]. In the present paper, we obtain the packing and open packing number of various snakes.

§2. Main Results

Theorem 2.1 *For $n \geq 3$, $\rho(G) = \left\lceil \frac{n}{3} \right\rceil$, where G is triangular snake T_n and double triangular snake $D(T_n)$.*

Proof The triangular snake T_n is obtained from a path P_n with vertices $v_1, v_2, \cdots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, 3, \cdots, n-1$ while to construct double triangular snake $D(T_n)$ from a path P_n with vertices $v_1, v_2, \cdots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, 3, \cdots, n-1$ and to a new vertex u_i for $i = 1, 2, \cdots, n-1$.

If S is any packing set of G then it is obvious that v_1 must in S as $d_G(v_1) = 2 = \delta(G)$.

We construct a set S of vertices as follows:

$$S = \left\{ v_{3i+1} / 0 \leq i \leq \left\lceil \frac{n}{3} \right\rceil - 1 \right\}$$

Then $|S| = \left\lceil \frac{n}{3} \right\rceil$. Moreover S is a packing set of G as $N[v] \cap N[u] \neq \phi$ for all $v, u \in S$. For any $w \in V(G) - S$, $N[v] \cap N[w] \neq \phi$ and $N[u] \cap N[w] \neq \phi$. Thus, S is a maximal packing set of G. Therefore any superset containing the vertices greater than that of $|S|$ can not be a packing set of G. Hence

$$\rho(G) = \left\lceil \frac{n}{3} \right\rceil. \qquad \square$$

Theorem 2.2 *For $n \geq 3$, $\rho^o(G) = \left\lceil \frac{n}{3} \right\rceil$, where G is triangular snake T_n and double triangular snake $D(T_n)$.*

Proof The triangular snake T_n is obtained from a path P_n with vertices $v_1, v_2, \cdots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, 3, \cdots, n-1$ while to construct double triangular snake $D(T_n)$ from a path P_n with vertices $v_1, v_2, \cdots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, 3, \cdots, n-1$ and to a new vertex u_i for $i = 1, 2, \cdots, n-1$.

If S is any open packing set of G then it is obvious that v_1 must in S as $d_G(v_1) = 2 = \delta(G)$.

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Then $|S| = \left\lceil \frac{n}{3} \right\rceil$. Moreover S is an open packing set of G as $N(v) \cap N(u) \neq \phi$ for all $v, u \in S$. For any $w \in V(G) - S$, $N(v) \cap N(w) \neq \phi$ and $N(u) \cap N(w) \neq \phi$. Thus, S is a maximal open packing set of G. Therefore any superset containing the vertices greater than that of $|S|$ can not be an open packing set of G. Hence

$$\rho^o(G) = \left\lceil \frac{n}{3} \right\rceil. \qquad \square$$

Illustration 2.3 The graph T_7 and its packing number and open packing number are shown Figure 1 while the graph $D(T_7)$ and its packing number and open packing number are shown in Figure 2.

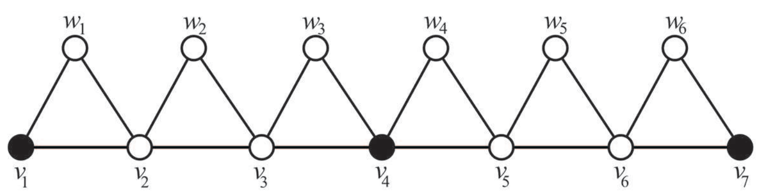

Figure 1 $\rho(T_7) = \rho^o(T_7) = 3$

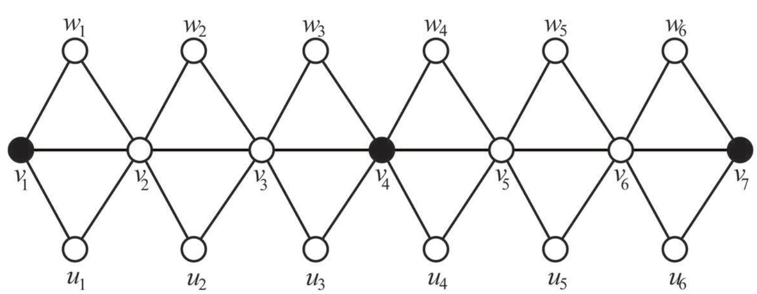

Figure 2 $\rho(D(T_7)) = \rho^o(D(T_7)) = 3$

Theorem 2.4 *For $n > 3$, $\rho(G) = \left\lceil \frac{n}{3} \right\rceil$, where G is alternate triangular snake AT_n and double alternate triangular snake $D(AT_n)$.*

Proof An alternate triangular snake AT_n is obtained from a path P_n with vertices $v_1, v_2, \cdots, v_n$ by joining v_i and v_{i+1} (alternately) to a new vertex w_i, $i = 1, 2, \cdots, n-1$ while to construct a double alternate triangular snake $D(AT_n)$ from a path P_n with vertices $v_1, v_2, \cdots, v_n$ by joining v_i and v_{i+1} (alternately) to a new vertex w_i, $i = 1, 2, \cdots, n-1$ and to a new vertex u_i for $i = 1, 2, \cdots, n-1$.

If S is any packing set of G then it is obvious that v_1 must in S as

$$d_G(v_1) = \delta(G) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

We construct a set S of vertices as follows:

$$S = \left\{ v_{3i+1} / 0 \le i \le \left\lceil \frac{n}{3} \right\rceil - 1 \right\}$$

Then $|S| = \left\lceil \frac{n}{3} \right\rceil$. Moreover S is a packing set of G as $N[v] \cap N[u] \neq \phi$ for all $v, u \in S$. For any $w \in V(G) - S$, $N[v] \cap N[w] \neq \phi$ and $N[u] \cap N[w] \neq \phi$. Thus, S is a maximal packing set of G. Therefore any superset containing the vertices greater than that of $|S|$ can not be a packing set of G. Hence

$$\rho(G) = \left\lceil \frac{n}{3} \right\rceil. \qquad \square$$

Illustration 2.5 The graph AT_7 and its packing number is shown Figure 3 while the graph $D(AT_7)$

and its packing number is shown in Figure 4.

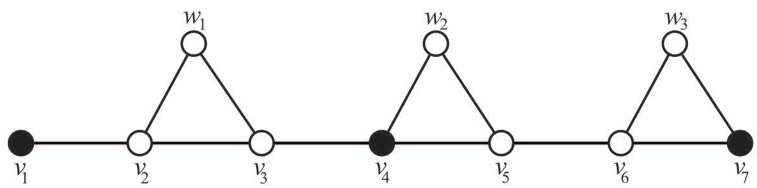

Figure 3 $\rho(AT_7) = 3$

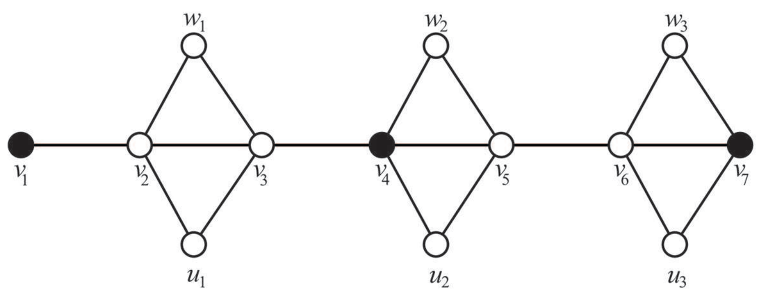

Figure 4 $\rho(D(AT_7)) = 3$

Theorem 2.6 *For $n > 3$, $\rho^o(G) = \left\lceil \dfrac{n}{2} \right\rceil$, where G is alternate triangular snake AT_n and double alternate triangular snake $D(AT_n)$.*

Proof An alternate triangular snake AT_n is obtained from a path P_n with vertices $v_1, v_2, \cdots, v_n$ by joining v_i and v_{i+1} (alternately) to a new vertex w_i, $i = 1, 2, \cdots, n-1$ while to construct a double alternate triangular snake $D(AT_n)$ from a path P_n with vertices $v_1, v_2, \cdots, v_n$ by joining v_i and v_{i+1} (alternately) to a new vertex w_i, $i = 1, 2, \cdots, n-1$ and to a new vertex u_i for $i = 1, 2, \cdots, n-1$.

If S is any open packing set of G then it is obvious that v_1 must in S as

$$d_G(v_1) = \delta(G) = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

We construct a set S of vertices as follows:

$$S = \begin{cases} \left\{ v_{4i+1}, v_{4i+2} / 0 \leq i \leq \left\lceil \dfrac{n}{5} \right\rceil \right\} & \text{for } n \text{ is odd} \\ \\ \left\{ v_{4i+2}, v_{4i+3} / 0 \leq i \leq \left\lfloor \dfrac{n}{5} \right\rfloor \right\} & \text{for } n \text{ is odd} \end{cases}$$

Then $|S| = \left\lceil \dfrac{n}{2} \right\rceil$. Moreover S is an open packing set of G as $N(v) \cap N(u) \neq \phi$ for all $v, u \in S$. For any $w \in V(G) - S$, $N(v) \cap N(w) \neq \phi$ and $N(u) \cap N(w) \neq \phi$. Thus, S is a maximal open packing set of G. Therefore any superset containing the vertices greater than that of $|S|$ can not be an open packing set of G. Hence

$$\rho^o(G) = \left\lceil \frac{n}{3} \right\rceil. \qquad \square$$

Illustration 2.7 The graph AT_7 and its open packing number is shown Figure 5 while the graph $D(AT_7)$ and its open packing number is shown in Figure 6.

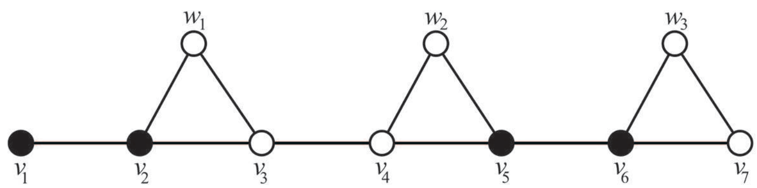

Figure 5 $\rho^o(AT_7) = 4$

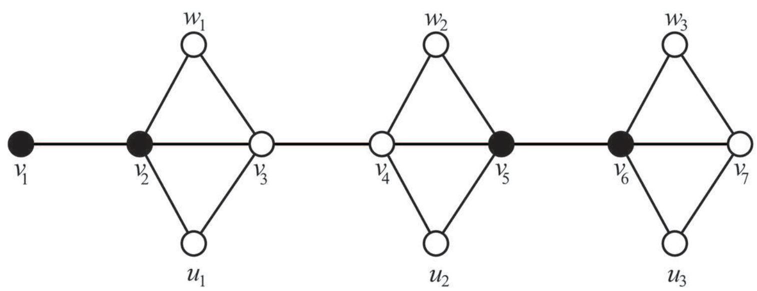

Figure 6 $\rho^o(D(AT_7)) = 4$

§3. Concluding Remarks

The concept of packing number relates three important graph parameters - neighborhood of a vertex, adjacency between two vertices and domination in graphs. We have investigated packing and open packing numbers of triangular snakes.

Acknowledgment

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Second Status Connectivity Indices and its Coindices of Composite Graphs

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Abstract: In this paper, we obtain the exact formulae for the second status connectivity indices and its coindices of some composite graphs such as Cartesian product, join and composition of two connected graphs.

Key Words: Wiener index, status connectivity index, composite graph.

AMS(2010): 05C12, 05C76.

§1. Introduction

A *topological index* is a mathematical measure which correlates to the chemical structures of any simple finite graph. They are invariant under the graph isomorphism. They play an important role in the study of $QSAR/QSPR$. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, nanoscience, biological and other properties of chemical compounds. Wiener index is the first distance-based topological index that were defined by Wiener [5]. For more details, see [9,10,11,12].

The *status* [2] of a vertex $v \in V(G)$ is defined as the sum of its distance from every other vertex in $V(G)$ and is denoted by $\sigma_G(v)$, that is, $\sigma_G(v) = \sum\limits_{u \in V(G)} d_G(u,v)$, where $d_G(u,v)$ is the distance between u and v in G. The status of vertex v is also called as *transmisson* of v [2].

The *Wiener index* $W(G)$ of a connected graph G is defined as the sum of the distances between all pairs of vertices of G, that is,

$$W(G) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u,v) = \frac{1}{2} \sum_{u \in V(G)} \sigma_G(v).$$

The *first Zagreb index* is defined as

$$M_1(G) = \sum_{u \in V(G)} (d_G(u))^2 = \sum_{uv \in E(G)} (d_G(u) + d_G(v))$$

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and the *second Zagreb index* is defined as

$$M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v).$$

The Zagreb indices are found to have applications in QSPR and QSAR studies as well, see [7]. The *first* and *second Zagreb coindices* were first introduced by Ashrafi et al. [8]. They are defined as follows:

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} (d_G(u) + d_G(v))$$

and the *second Zagreb index* is defined as

$$\overline{M}_2(G) = \sum_{uv \notin E(G)} d_G(u)d_G(v).$$

Motivated by the invariants like Zagreb indices, Ramane et al.[1] proposed the *first status connectivity index* $S_1(G)$ and *first status connectivity coindex* $\overline{S}_1(G)$ of a connected graph G as

$$S_1(G) = \sum_{uv \in E(G)} \Big(\sigma_G(u) + \sigma_G(v)\Big) \text{ and } \overline{S}_1(G) = \sum_{uv \notin E(G)} \Big(\sigma_G(u) + \sigma_G(v)\Big).$$

Similarly, the *second status connectivity index* $S_2(G)$ and *second status connectivity coindex* $\overline{S}_2(G)$ of a connected graph G as

$$S_2(G) = \sum_{uv \in E(G)} \sigma_G(u)\sigma_G(v) \text{ and } \overline{S}_2(G) = \sum_{uv \notin E(G)} \sigma_G(u)\sigma_G(v).$$

The bounds for the status connectivity indices are determined in [1]. Also they are discussed the linear regression analysis of the distance-based indices with the boiling points of benzenoid hydrocarbons and the linear model based on the status index is better than the models corresponding to the other distance based indices. In this sequence, here we obtain the exact formulae for second status connectivity indices and its coindices of some composite graphs such as Cartesian product, join, composition of two connected graphs.

§2. Main Results

In this section, we obtain the second status connectivity indices and its coindices of Cartesian product, join and composition of two graphs.

Lemma 2.1 *Let G be a connected graph on n vertices. Then*

$$\overline{S}_2(G) = 2(W(G)) - \frac{1}{2} \sum_{u \in V(G)} (\sigma_G(u))^2 - S_2(G).$$

Proof By the definition of $\overline{S}_2$, we obtain:

$$\begin{aligned}
\overline{S}_2(G) &= \sum_{uv\notin E(G)} \sigma_G(u)\sigma_G(v)\\
&= \sum_{\{u,v\}\subseteq V(G)} \sigma_G(u)\sigma_G(v) - \sum_{uv\in E(G)} \sigma_G(u)\sigma_G(v)\\
&= \frac{1}{2}\Bigg(\Big(\sum_{u\in V(G)} \sigma_G(u)\Big)^2 - \sum_{u\in V(G)} (\sigma_G(u))^2\Bigg) - S_2(G)\\
&= 2(W(G)) - \frac{1}{2}\sum_{u\in V(G)} (\sigma_G(u))^2 - S_2(G).
\end{aligned}$$

□

Let C_n and P_n denote the cycle and path on n vertices, respectively. It is known that [1]

$$S_1(P_n) = \frac{1}{3}n(n-1)(2n-1) \quad \text{and} \quad W(P_n) = \frac{n(n^2-1)}{6}$$

and

$$S_1(C_n) = \begin{cases} \frac{n^3}{2}, & \text{if } n \text{ is even,}\\ \frac{n(n^2-1)}{2}, & \text{otherwise;}\end{cases} \quad \text{and} \quad W(C_n) = \begin{cases} \frac{n^3}{8}, & \text{if } n \text{ is even,}\\ \frac{n(n^2-1)}{8}, & \text{otherwise.}\end{cases}$$

We therefore have that

Lemma 2.2 *For cycle C_n and path P_n, we get that*

(1) *For $n \geq 3$, $S_2(C_n) = \begin{cases} \frac{n^5}{16} & \text{if } n \text{ is even}\\ \frac{n(n^2-1)^2}{16} & \text{if } n \text{ is odd;}\end{cases}$*

(2) $S_2(P_n) = \frac{n^2(n-1)}{4}$.

2.1 Cartesian Product

The *Cartesian product,* $G \,\square\, H$, of the graphs G and H has the vertex set $V(G \,\square\, H) = V(G) \times V(H)$ and $(u,x)(v,y)$ is an edge of $G \,\square\, H$ if $u = v$ and $xy \in E(H)$ or, $uv \in E(G)$ and $x = y$. To each vertex $u \in V(G)$, there is an isomorphic copy of H in $G \,\square\, H$ and to each vertex $v \in V(H)$, there is an isomorphic copy of G in $G \,\square\, H$.

Theorem 2.3 *Let G and H be two connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$\begin{aligned}
S_2(G\square H) &= n_2^3 S_2(G) + n_1^3 S_2(H) + 2n_1 n_2 (S_1(G)W(H) + S_1(H)W(G))\\
&\quad + n_2^2 m_2 \sum_{u_i\in V(G)} (\sigma_G(u_i))^2 + n_1^2 m_1 \sum_{v_s\in V(H)} (\sigma_H(v_s))^2.
\end{aligned}$$

Proof From the structure of $G\square H$, the distance between two vertices (u_i, v_r) and (u_k, v_s) of $G\square H$ is $d_G(u_i, u_k) + d_H(v_r, v_s)$. Moreover, the degree of a vertex (u_i, v_r) in $V(G\square H)$ is $d_G(u_i) + d_H(v_r)$. By the definition of $\sigma(u)$ for the graph $G\Box H$ and a vertex $(u_i, v_r) \in V(G\Box H)$, we have

$$\begin{aligned}\sigma_{G\Box H}((u_i,v_r)) &= \sum_{(u_k,v_s)\in V(G\Box H)} d_{G\Box H}((u_i,v_r),(u_k,v_s))\\ &= \sum_{u_k\in V(G)}\sum_{v_s\in V(H)}\Big(d_G(u_i,u_k)+d_G(v_r,v_s)\Big)\\ &= n_2\sigma_G(u_i)+n_1\sigma_H(v_r). \qquad (2.1)\end{aligned}$$

Hence by the definitions of S_2 and $G\Box H$, we have

$$\begin{aligned}S_2(G\Box H) &= \sum_{(u_i,v_s)(u_k,v_s)\in E(G\Box H)} \sigma_{G\Box H}((u_i,v_s))\sigma_{G\Box H}((u_k,v_s))\\ &\quad+\sum_{(u_i,v_s)(u_k,v_s)\in E(G\Box H)} \sigma_{G\Box H}((u_i,v_r))\sigma_{G\Box H}((u_i,v_s))\\ &= A_1+A_2, \qquad (2.2)\end{aligned}$$

where

$$\begin{aligned}A_1 &= \sum_{(u_i,v_s)(u_k,v_s)\in E(G\Box H)} \sigma_{G\Box H}((u_i,v_s))\sigma_{G\Box H}((u_k,v_s))\\ &= \sum_{u_iu_k\in E(G)}\sum_{v_s\in V(H)}\Big(n_2\sigma_G(u_i)+n_1\sigma_H(v_s)\Big)\Big(n_2\sigma_G(u_k)+n_1\sigma_H(v_s)\Big),\ \text{by}\ \ (2.1)\\ &= \sum_{u_iu_k\in E(G)}\sum_{v_s\in V(H)}\Big(n_2^2\sigma_G(u_i)\sigma_G(u_k)+n_1n_2\sigma_G(u_i)\sigma_H(v_s)\\ &\quad+n_1n_2\sigma_H(v_s)\sigma_G(u_k)+n_1^2(\sigma_H(v_s))^2\Big)\\ &= n_2^3\sum_{u_iu_k\in E(G)}\sigma_G(u_i)\sigma_G(u_k)+n_1n_2\sum_{v_s\in V(H)}\sigma_H(v_s)\sum_{u_iu_k\in E(G)}(\sigma_G(u_i)+\sigma_G(u_k))\\ &\quad+n_1^2m_1\sum_{v_s\in V(H)}(\sigma_H(v_s))^2\\ &= n_2^3S_2(G)+2n_1n_2S_1(G)W(H)+n_1^2m_1\sum_{v_s\in V(H)}(\sigma_H(v_s))^2.\end{aligned}$$

and a similar argument of A_1, we obtain

$$\begin{aligned}A_2 &= \sum_{(u_i,v_s)(u_k,v_s)\in E(G\Box H)} \sigma_{G\Box H}((u_i,v_r))\sigma_{G\Box H}((u_i,v_s))\\ &= n_1^3S_2(H)+2n_1n_2S_1(H)W(G)+n_2^2m_2\sum_{u_i\in V(G)}(\sigma_G(u_i))^2.\end{aligned}$$

From (2.2) and A_1, A_2, we obtain:

$$\begin{aligned}S_2(G\Box H) &= n_2^3S_2(G)+n_1^3S_2(H)+2n_1n_2(S_1(G)W(H)+S_1(H)W(G))\\ &\quad+n_2^2m_2\sum_{u_i\in V(G)}(\sigma_G(u_i))^2+n_1^2m_1\sum_{v_s\in V(H)}(\sigma_H(v_s))^2. \qquad \Box\end{aligned}$$

Remark 2.4 For each vertex (u_i, v_r) in $G\Box H$,

$$\begin{aligned}
&\sum_{(u_i,v_r)\in V(G\Box H)} \Big(\sigma_{G\Box H}((u_i,v_r))\Big)^2 \\
&= \sum_{u_i\in V(G)}\sum_{v_r\in V(H)} \Big(n_2\sigma_G(u_i)+n_1\sigma_H(v_r)\Big)^2, \text{ by } (2.1) \\
&= \sum_{u_i\in V(G)}\sum_{v_r\in V(H)} \Big(n_2^2(\sigma_G(u_i))^2+n_1^2(\sigma_H(v_r))^2+2n_1n_2\sigma_G(u_i)\sigma_H(v_r)\Big) \\
&= n_2^3\sum_{u_i\in V(G)}(\sigma_G(u_i))^2+n_1^3\sum_{v_r\in V(H)}(\sigma_H(v_r))^2+8n_1n_2W(G)W(H).
\end{aligned}$$

By Theorem 2.3, Lemma 2.1, Remark 2.4 and this fact that [3], $W(G\Box H)=n_2^2W(G)+n_1^2W(H)$, the following theorem is straightforward.

Theorem 2.5 *Let G and H be two connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$\begin{aligned}
\overline{S}_2(G\Box H) &= 2[n_2^2W(G)+n_1^2W(H)]^2-n_2^3S_2(G)-n_1^3S_2(H) \\
&\quad -2n_1n_2[S_1(G)W(H)+S_1(H)W(G)+2W(G)W(H)] \\
&\quad -\frac{n_2^2(n_2+2m_2)}{2}\sum_{u_i\in V(G)}(\sigma_G(u_i))^2-\frac{n_1^2(n_1+2m_1)}{2}\sum_{v_r\in V(H)}(\sigma_H(v_r))^2.
\end{aligned}$$

2.2 Join

The *join* $G+H$ of two graphs G and H is the union $G\cup H$ together with all the edges joining $V(G)$ and $V(H)$. From the structure of $G+H$, the distance between two vertices u and v of $G+H$ is

$$d_{G+H}(u,v)=\begin{cases}0, \text{ if } u=v,\\ 1, \text{ if } uv\in E(G) \text{ or } uv\in E(H) \text{ or } (u\in V(G) \text{ and } v\in V(H)),\\ 2, \text{ otherwise.}\end{cases}$$

Moreover, the degree of a vertex v in $V(G+H)$ is

$$d_{G+H}(v)=\begin{cases}d_G(v)+|V(H)|, if \;\; v\in V(G),\\ d_H(v)+|V(G)|, if \;\; v\in V(H).\end{cases}$$

Theorem 2.6 *Let G and H be two connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$\begin{aligned}
S_2(G+H) &= M_2(G)+M_2(H)-(2n_1+n_2-2)M_1(G) \\
&\quad -(2n_2+n_1-2)M_1(H) \\
&\quad +(2n_1+n_2-2)[(2n_1+n_2-2)m_1-2n_1m_2] \\
&\quad -(2n_2+n_1-2)[(2n_2+n_1-2)m_2-2n_2m_1 \\
&\quad +n_1n_2(2n_1+n_2-2)]+4m_1m_2.
\end{aligned}$$

Proof Let u be a vertex in $V(G)$. Then from the structure of $G+H$, we obtain:

$$\begin{aligned}\sigma_{G+H}(u) &= \sum_{v\in V(G+H)} d_{G+H}((u,v)) \\ &= \sum_{v\in V(G)\ u\neq v, uv\notin E(G)} 2 + \sum_{v\in V(G),\ u\neq v, uv\in E(G)} 1 + \sum_{v\in V(H)} 1 \\ &= 2n_1+n_2-2-d_G(u).\end{aligned}$$

Similarly, if v is a vertex of H, then $\sigma_{G+H}(v)=2n_2+n_1-2-d_G(v)$.

The edge set of $G+H$ can be partitioned into three subsets, namely,

$E_1=\{uv\in E(G+H)|uv\in E(G)\}$,

$E_2=\{uv\in E(G+H)|uv\in E(H)\}$ and

$E_3=\{uv\in E(G+H)|u\in V(G),\ v\in V(H)\}$.

The contribution of the edges in E_1 is given by

$$\begin{aligned}S_2(G+H) &= \sum_{uv\in E_1}\sigma_{G+H}(u)\sigma_{G+H}(v) \\ &= \sum_{uv\in E(G)}\Big(2n_1+n_2-2-d_G(u)\Big)\Big(2n_1+n_2-2-d_G(v)\Big) \\ &= \sum_{uv\in E(G)}\big[(2n_1+n_2-2)^2-(2n_1+n_2-2)d_G(v) \\ &\quad -(2n_1+n_2-2)d_G(u)+d_G(u)d_G(v)] \\ &= (2n_1+n_2-2)^2m_1-(2n_1+n_2-2)M_1(G)+M_2(G). \qquad (2.3)\end{aligned}$$

Similarly, the contribution of the edges in E_2 is given by

$$\begin{aligned}S_2(G+H) &= \sum_{uv\in E_2}\sigma_{G+H}(u)\sigma_{G+H}(v) \\ &= (2n_2+n_1-2)^2m_2-(2n_2+n_1-2)M_1(H)+M_2(H). \qquad (2.4)\end{aligned}$$

The contribution of the edges in E_3 is given by

$$\begin{aligned}S_2(G+H) &= \sum_{uv\in E_3}\sigma_{G+H}(u)\sigma_{G+H}(v) \\ &= \sum_{u\in V(G)}\sum_{v\in V(H)}\Big(2n_1+n_2-2-d_G(u)\Big)\Big(2n_2+n_1-2-d_H(v)\Big) \\ &= \sum_{u\in V(G)}\sum_{v\in V(H)}\Big[(2n_1+n_2-2)(2n_2+n_1-2)-(2n_1+n_2-2)d_H(v) \\ &\quad -(2n_2+n_1-2)d_G(u)+d_G(u)d_H(v)\Big] \\ &= (2n_1+n_2-2)(2n_2+n_1-2)n_1n_2-2n_1m_2(2n_1+n_2-2) \\ &\quad -2n_2m_1(2n_2+n_1-2)+4m_1m_2. \qquad (2.5)\end{aligned}$$

The total contribution of the edges in $G+H$ and its $S_2(G+H)$ is given by

$$\begin{aligned} S_2(G+H) &= M_2(G)+M_2(H)-(2n_1+n_2-2)M_1(G) \\ &\quad -(2n_2+n_1-2)M_1(H) \\ &\quad +(2n_1+n_2-2)[(2n_1+n_2-2)m_1-2n_1m_2] \\ &\quad -(2n_2+n_1-2)[(2n_2+n_1-2)m_2 \\ &\quad -2n_2m_1+n_1n_2(2n_1+n_2-2)]+4m_1m_2. \end{aligned}$$ □

Remark 2.7 For each vertex v in $G+H$,

$$\begin{aligned} \sum_{v\in V(G+H)}(\sigma_{G+H}(v))^2 &= \sum_{v\in V(G)}(\sigma_{G+H}(v))^2+\sum_{v\in V(H)}(\sigma_{G+H}(v))^2 \\ &= \sum_{v\in V(G)}(2n_1+n_2-2-d_G(u))^2+\sum_{v\in V(H)}(2n_2+n_1-2-d_G(v))^2 \\ &= \sum_{v\in V(G)}\Big((2n_1+n_2-2)^2+(d_G(v))^2-2(2n_1+n_2-2)d_G(v)\Big) \\ &\quad +\sum_{v\in V(H)}\Big((2n_2+n_1-2)^2+(d_H(v))^2-2(2n_2+n_1-2)d_H(v)\Big) \\ &= (2n_1+n_2-2)^2n_1+M_1(G)-4m_1(2n_1+n_2-2) \\ &\quad +(2n_2+n_1-2)^2n_2+M_1(H)-4m_2(2n_2+n_1-2). \end{aligned}$$

According to [3], we know that

$$\begin{aligned} W(G+H) &= |V(G)|\,(|V(G)|-1)+|V(H)|\,(|V(H)|-1) \\ &\qquad +|V(G)|\,|V(H)|-|E(G)|-|E(H)|\,. \end{aligned}$$

By this formula, Theorem 2.6, Lemma 2.1 and Remark 2.7, we obtain the following theorem.

Theorem 2.8 *Let G and H be two connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$\begin{aligned} \overline{S}_2(G+H) &= \frac{M_1(G)}{2}\Big(4n_1+2n_2-5\Big)+\frac{M_1(H)}{2}\Big(4n_2+2n_1-5\Big) \\ &\quad -M_2(G)-M_2(H)+2\Big(n_1(n_1-1)+n_2(n_2-1)+n_1n_2-m_1-m_2\Big) \\ &\quad -(2n_1+n_2-2)\Big((2n_1+n_2-2)(\frac{n_1}{2}+m_1)-2(m_1+n_1m_2)\Big) \\ &\quad -(2n_2+n_1-2)\Big((2n_2+n_1-2)(\frac{n_2}{2}-m_2)-2(m_2-n_2m_1) \\ &\quad -n_1n_2(2n_1+n_2-2)\Big)-4m_1m_2. \end{aligned}$$

2.3 Composition

The *composition* of two graphs G and H is denoted by $G[H]$. The vertex set of $G[H]$ is $V(G)\times V(H)$ and any two vertices (u_i, v_r) and (u_k, v_s) are adjacent if and only if $u_iu_k \in E(G)$ or $u_i = u_k$ and $v_rv_s \in E(H)$.

Theorem 2.9 *Let G and H be two connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively.*

Then

$$\begin{aligned} S_2(G[H]) &= n_2^4 S_2(G) + 2n_2^2(n_2(n_2-1)-m_2)S_1(G) + 8n_2m_2(n_2-1)W(G) \\ &\quad -2n_2W(G)M_1(H) - 2n_1(n_2-1)M_1(H) + n_1M_2(H) \\ &\quad +n_2^2 m_2 \sum_{u_i \in V(G)} (\sigma_G(u_i))^2 + 4(n_2-1)^2(n_1m_2 + m_1n_2^2) \\ &\quad +4m_1m_2(m_2 - 2n_2(n_2-1)). \end{aligned}$$

Proof For the composition of two graphs, the degree of a vertex (u,v) of $G[H]$ is given by $d_{G[H]}((u,v)) = n_2 d_G(u) + d_H(v)$. Moreover, the distance between two vertices (u_i, v_r) and (u_k, v_s) of $G[H]$ is

$$d_{G[H]}((u_i,v_r),(u_k,v_s)) = \begin{cases} d_G(u_i,u_k) & u_i \neq u_k \\ 2 & u_i = u_k,\ v_rv_s \notin E(H) \\ 1 & u_i = u_k,\ v_rv_s \in E(H). \end{cases}$$

Let (u_i, v_r) be a vertex of $G[H]$. Then

$$\begin{aligned} \sigma_{G[H]}((u_i,v_r)) &= \sum_{(u_k,v_s)\in V(G[H])} d_{G[H]}((u_i,v_r),(u_k,v_s)) \\ &= \sum_{(u_k,v_s)\in V(G[H]),\ u_i\neq u_k} d_G(u_i,u_k) + \sum_{(u_i,v_s)\in V(G[H])} d_{G[H]}((u_i,v_r),(u_i,v_s)) \\ &= n_2\sigma_G(u_i) + d_H(v_r) + 2(n_2 - 1 - d_H(v_r)) \\ &= n_2\sigma_G(u_i) + 2(n_2-1) - d_H(v_r). \end{aligned} \tag{2.6}$$

From the structure of $G[H]$ and definition of S_2, we have

$$\begin{aligned} S_2(G[H]) &= \sum_{u_i\in V(G)} \sum_{v_rv_s\in E(H)} \sigma_{G[H]}((u_i,v_r))\sigma_{G[H]}((u_i,v_s)) \\ &\quad + \sum_{u_iu_k\in E(G)} \sum_{v_r\in V(H)} \sum_{v_s\in V(H)} \sigma_{G[H]}((u_i,v_r))\sigma_{G[H]}((u_i,v_s)) \\ &= A_1 + A_2, \end{aligned} \tag{2.7}$$

where,

$$\begin{aligned} A_1 &= \sum_{u_i\in V(G)} \sum_{v_rv_s\in E(H)} \sigma_{G[H]}((u_i,v_r))\sigma_{G[H]}((u_i,v_s)) \\ &= \sum_{u_i\in V(G)} \sum_{v_rv_s\in E(H)} \Big(n_2\sigma_G(u_i) + 2(n_2-1) - d_H(v_r)\Big)\Big(n_2\sigma_G(u_i) + 2(n_2-1) - d_H(v_s)\Big) \\ &= \sum_{u_i\in V(G)} \sum_{v_rv_s\in E(H)} \Big[n_2^2(\sigma_G(u_i))^2 + 2(n_2-1)n_2\sigma_G(u_i) - n_2\sigma_G(u_i)d_H(v_s) + 2(n_2-1)n_2\sigma_G(u_i) \\ &\quad +4(n_2-1)^2 - 2(n_2-1)d_H(v_s) - n_2\sigma_G(u_i)d_H(v_r) - 2(n_2-1)d_H(v_r) + d_H(v_r)d_H(v_s)\Big] \\ &= \sum_{u_i\in V(G)} \sum_{v_rv_s\in E(H)} \Big[n_2^2(\sigma_G(u_i))^2 + 4n_2(n_2-1)\sigma_G(u_i) + 4(n_2-1) - n_2\sigma_G(u_i)(d_H(v_r)+d_H(v_s)) \\ &\quad -2(n_2-1)(d_H(v_r)+d_H(v_s)) + d_H(v_r)d_H(v_s)\Big] \\ &= n_2^2 m_2 \sum_{u_i\in V(G)} (\sigma_G(u_i))^2 + 8n_2(n_2-1)m_2W(G) + n_1M_2(H) \end{aligned}$$

$$-2n_2W(G)M_1(H) - 2(n_2-1)n_1M_1(H).$$

$$\begin{aligned}
A_2 &= \sum_{u_iu_k\in V(G)}\sum_{v_r\in V(H)}\sum_{v_s\in V(H)} \sigma_{G[H]}((u_i,v_r))\sigma_{G[H]}((u_k,v_s)) \\
&= \sum_{u_iu_k\in V(G)}\sum_{v_r\in V(H)}\sum_{v_s\in V(H)} \Big(n_2\sigma_G(u_i)+2(n_2-1)-d_H(v_r)\Big)\Big(n_2\sigma_G(u_k)+2(n_2-1)-d_H(v_s)\Big) \\
&= \sum_{u_iu_k\in V(G)}\sum_{v_r\in V(H)}\sum_{v_s\in V(H)} \Big[n_2^2\sigma_G(u_i)\sigma_G(u_k)+2(n_2-1)n_2(\sigma_G(u_i)+\sigma_G(u_k))+4(n_2-1)^2 \\
&\quad -n_2\sigma_G(u_i)d_H(v_s)-n_2d_H(v_r)\sigma_G(u_k)-2(n_2-1)(d_H(v_r)+d_H(v_r))+d_H(v_r)d_H(v_s)\Big] \\
&= n_2^4S_2(G)+2n_2^2(n_2(n_2-1)-m_2)S_1(G)-8n_2m_1m_2(n_2-1)+4m_1m_2^2+4(n_2-1)^2m_1n_2^2.
\end{aligned}$$

Hence

$$\begin{aligned}
S_2(G[H]) &= n_2^4S_2(G)+2n_2^2(n_2(n_2-1)-m_2)S_1(G)+8n_2m_2(n_2-1)W(G) \\
&\quad -2n_2W(G)M_1(H)-2n_1(n_2-1)M_1(H)+n_1M_2(H) \\
&\quad +n_2^2m_2\sum_{u_i\in V(G)}(\sigma_G(u_i))^2+4(n_2-1)^2(n_1m_2+m_1n_2^2) \\
&\quad +4m_1m_2(m_2-2n_2(n_2-1)). \qquad \square
\end{aligned}$$

Remark 2.10 Let (u_i,v_r) be a vertex of $G[H]$. Then

$$\begin{aligned}
\sum_{(u_i,v_r)\in V(G[H])}(\sigma_{G[H]}((u_i,v_r)))^2 &= \sum_{u_i\in V(G)}\sum_{v_r\in V(H)}(n_2\sigma_G(u_i)+2(n_2-1)-d_H(v_r))^2 \\
&= \sum_{u_i\in V(G)}\sum_{v_r\in V(H)}\Big(n_2^2(\sigma_G(u_i))^2+4(n_2-1)^2+(d_H(v_r))^2 \\
&\quad +4n_2(n_2-1)\sigma_G(u_i)-2n_2\sigma_G(u_i)d_H(v_r)-2(n_2-1)d_H(v_r)\Big) \\
&= n_2^3\sum_{u_i\in V(G)}(\sigma_G(u_i))^2+n_1M_1(H) \\
&\quad +4n_2(n_2(n_2-1)-m_2)\sum_{u_i\in V(G)}\sigma_G(u_i) \\
&\quad +4(n_2-1)(n_1n_2(n_2-1)-m_2).
\end{aligned}$$

Recall from [3] that

$$W(G[H]) = |V(H)|^2\,(W(G)+|V(G)|)-|V(G)|\,(|V(H)|+|E(H)|).$$

In the next theorem, we obtain a formula for $\overline{S}_1(G[H])$ according to $W(G[H])$, $S_2(G[H])$ and Remark 2.10.

Theorem 2.11 *Let G and H be two connected graphs with n_1,n_2 vertices and m_1,m_2 edges, respectively. Then*

$$\begin{aligned}
\overline{S}_2(G[H]) &= \Big(2n_2W(G)+2n_1(n_2-1)-\frac{n_1}{2}\Big)M_1(H)-n_1M_2(H)-n_2^2S_2(G) \\
&\quad -2n_2^2(n_2(n_2-1)-m_2)S_1(G)-\Big(8n_2m_2(n_1-1)+2n_2^2\Big)W(G)
\end{aligned}$$

$$-\frac{n_2^2}{2}(n_2-2m_2)\sum_{u_i\in V(G)}(\sigma_G(u_i))^2-2n_2(n_2(n_2-1)-m_2)\sum_{u_i\in V(G)}\sigma_G(u_i)$$
$$+n_1n_2(2n_2-1)-n_1m_2-2(n_2-1)^2(n_1n_2+2n_1m_2+2m_1n_2^2)$$
$$+2m_2(n_2-1)(4m_1n_2+1)-4m_1m_2^2.$$

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A Note on Detour Radial Signed Graphs

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Abstract: In this paper we introduced a new notion detour radial signed graph of a signed graph and its properties are obtained. Also, we obtained the structural characterization of detour radial signed graphs. Further, we presented some switching equivalent characterizations.

Key Words: Signed graphs, balance, switching, detour radial signed graph, radial signed graph.

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§1. Introduction

For standard terminology and notion in graph theory, we refer the reader to the text-book of Harary [2]. The non-standard will be given in this paper as and when required.

Let $G = (V, E)$ be a connected graph.For any two vertices $u, v \in V(G)$, the detour distance $D(u, v)$ is the length of the longest $u - v$ path in G. The eccentricity $e(u)$ of a vertex u is the distance to a vertex farthest from u. The radius $r(G)$ of G is defined by

$$r(G) = \min\{e(u) : u \in G\}.$$

For any vertex u in G, the detour eccentricity $D_e(u)$ of u is the detour distance to a vertex farthest from u. The detour radius $D_r(G)$ of G is defined by $D_r(G) = \min\{D_e(u) : u \in G\}$. The diameter $d(G)$ of G is defined by $d(G) = \max\{e(u) : u \in G\}$ and the detour diameter $D_d(G)$ of G is $\max\{D_e(u) : u \in G\}$.

The detour radial graph $\mathcal{DR}(G)$ of $G = (V, E)$ is a graph with $V(\mathcal{DR}(G)) = V(G)$ and any two vertices u and v in $\mathcal{DR}(G)$ are joined by an edge if and only if $D(u, v) = D_r(G)$. This concept were introduced by Ganeshwari and Pethanachi Selvam [1].

To model individuals' preferences towards each other in a group, Harary [3] introduced the concept of signed graphs in 1953. A signed graph $S = (G, \sigma)$ is a graph $G = (V, E)$ whose edges are labeled with positive and negative signs (i.e., $\sigma : E(G) \to \{+, -\}$). The vertexes of a graph represent people and an edge connecting two nodes signifies a relationship between individuals. The signed graph captures the attitudes between people, where a positive (negative edge) represents liking (disliking). An unsigned graph is a signed graph with the signs removed. Similar to an unsigned graph, there are many active

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areas of research for signed graphs. For more new notions on signed graphs refer the papers.

The sign of a cycle (this is the edge set of a simple cycle) is defined to be the product of the signs of its edges; in other words, a cycle is positive if it contains an even number of negative edges and negative if it contains an odd number of negative edges. A signed graph S is said to be balanced if every cycle in it is positive. A signed graph S is called totally unbalanced if every cycle in S is negative. A chord is an edge joining two non adjacent vertices in a cycle.

A *marking* of S is a function $\zeta : V(G) \to \{+, -\}$. Given a signed graph S one can easily define a marking ζ of S as follows:

For any vertex $v \in V(S)$,

$$\zeta(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking ζ of S is called canonical marking *of S.*

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Theorem 1.1 *A signed graph S is balanced if and only if either of the following equivalent conditions is satisfied:*

(i) Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2; (Harary [3])

(ii) There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. (Sampathkumar [4])

Switching S with respect to a marking ζ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The resulting signed graph $S_\zeta(S)$ is said switched signed graph. A signed graph S is called to switch to another signed graph S' written $S \sim S'$, whenever their exists a marking ζ such that $S_\zeta(S) \cong S'$, where $\cong$ denotes the usual equivalence relation of isomorphism in the class of signed graphs. Hence, if $S \sim S'$, we shall say that S and S' are switching equivalent. Two signed graphs S_1 and S_2 are signed isomorphic (written $S_1 \cong S_2$) if there is a one-to-one correspondence between their vertex sets which preserve adjacency as well as sign.

Two signed graphs $S_1 = (G_1, \sigma_1)$ and $S_2 = (G_2, \sigma_2)$ are said to be *weakly isomorphic* (see [21]) or *cycle isomorphic* (see [22]) if there exists an isomorphism $\phi : G_1 \to G_2$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. More results on signed graphs can be found in references [4-22]. For example, the following result is well known.

Theorem 1.2 (T. Zaslavsky, [22]) *Given a graph G, any two signed graphs in $\psi(G)$, where $\psi(G)$ denotes the set of all the signed graphs possible for a graph G, are switching equivalent if and only if they are cycle isomorphic.*

§2. **Detour Radial Signed Graphs**

Motivated by the existing definition of complement of a signed graph, we now extend the notion of detour radial graphs to signed graphs as follows: The *detour radial signed graph* $\mathcal{DR}(S)$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $\mathcal{DR}(G)$ and sign of any edge uv is $\mathcal{DR}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called detour radial signed graph, if $S \cong \mathcal{DR}(S')$ for some signed graph S'. The following result restricts the class of detour radial graphs.

Theorem 2.1 *For any signed graph $S = (G, \sigma)$, its detour radial signed graph $\mathcal{DR}(S)$ is balanced.*

Proof Since sign of any edge $e = uv$ in $\mathcal{DR}(S)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of S, by Theorem 1.1, $\mathcal{DR}(S)$ is balanced. □

For any positive integer k, the k^{th} iterated detour radial signed graph, $\mathcal{DK}^k(S)$ of S is defined as follows:

$$\mathcal{DR}^0(S) = S, \; \mathcal{DR}^k(S) = \mathcal{DR}(\mathcal{DR}^{k-1}(S)).$$

Corollary 2.2 *For any signed graph $S = (G, \sigma)$ and for any positive integer k, $\mathcal{DR}^k(S)$ is balanced.*

The following result characterize signed graphs which are detour radial signed graphs.

Theorem 2.3 *A signed graph $S = (G, \sigma)$ is a detour radial signed graph if, and only if, S is balanced signed graph and its underlying graph G is a detour radial graph.*

Proof Suppose that S is balanced and G is a detour radial graph. Then there exists a graph G' such that $\mathcal{DR}(G') \cong G$. Since S is balanced, by Theorem 1.1, there exists a marking ζ of G such that each edge uv in S satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the signed graph $S' = (G', \sigma')$, where for any edge e in G', $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $\mathcal{DR}(S') \cong S$. Hence S is a detour radial signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a detour radial signed graph. Then there exists a signed graph $S' = (G', \sigma')$ such that $\mathcal{DR}(S') \cong S$. Hence, G is the detour radial graph of G' and by Theorem 2.1, S is balanced. □

In [1], the authors characterizes the graphs $G = (V, E)$ such that $G \cong \mathcal{DR}(G)$.

Theorem 2.4 *Let $G = (V, E)$ be a graph with atleast one cycle which covers all the vertices of G. Then G and the detour radial graph $\mathcal{DR}(G)$ are isomorphic if and only if G is isomorphic to either K_n or C_n or $K_{m,n}$ with $m = n$.*

In view of the above result, we now characterize the signed graphs such that the detour radial signed graph and its corresponding signed graph are switching equivalent.

Theorem 2.5 *For any signed graph $S = (G, \sigma)$ and its underlying graph G contains atleast one cycle which covers all the vertices. Then S and the detour radial signed graph $\mathcal{DR}(S)$ are cycle isomorphic if and only if the underlying of S satisfies the conditions of Theorem 2.4 and S is balanced.*

Proof Suppose $\mathcal{RD}(S) \sim S$. This implies, $\mathcal{DR}(G) \cong G$ and hence by Theorem 2.4, we see that the graph G satisfies the conditions in Theorem 2.4. Now, if S is any signed graph with underlying graph contains at least one Hamilton cycle and satisfies the conditions of Theorem 2.4. Then $\mathcal{DR}(S)$ is balanced and hence if S is unbalanced and its detour radial signed graph $\mathcal{DR}(S)$ being balanced can not be switching equivalent to S in accordance with Theorem 1.2. Therefore, S must be balanced.

Conversely, suppose that S balanced signed graph with the underlying graph G satisfies the conditions of Theorem 2.4. Then, since $\mathcal{DR}(S)$ is balanced as per Theorem 2.1 and since $\mathcal{DR}(G) \cong G$ by Theorem 2.4, the result follows from Theorem 1.2 again. □

In [5], P.S.K.Reddy introduced the notion radial signed graph of a signed graph and proved some results.

Theorem 2.6 *For any signed graph $S = (G, \sigma)$, its radial signed graph $\mathcal{R}(S)$ is balanced.*

In [1], the authors remarked that $\mathcal{DR}(G)$ and $\mathcal{R}(G)$ are isomorphic, if G is any cycle of odd length. We now characterize the signed graphs S such that $\mathcal{DR}(S) \sim \mathcal{R}(S)$.

Theorem 2.7 *For any signed graph $S = (G, \sigma)$, $\mathcal{DR}(S) \sim \mathcal{R}(S)$ if, and only if, $G \cong C_n$, where n is odd.*

Proof Suppose that $\mathcal{DR}(S) \sim \mathcal{R}(S)$. Then clearly, $\mathcal{DR}(G) \sim \mathcal{R}(G)$. Hence, G is any cycle of odd length.

Conversely, suppose that S is a signed graph whose underlying graph G is C_n, where n is odd. Then, $\mathcal{DR}(G) \cong \mathcal{R}(G)$. Since for any signed graph S, both $\mathcal{DR}(S)$ and $\mathcal{R}(S)$ are balanced, the result follows by Theorem 1.2. □

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A Note on 3-Remainder Cordial Labeling Graphs

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Abstract: Let G be a (p, q) graph. Let f be a function from $V(G)$ to the set $\{1, 2, \cdots, k\}$ where k is an integer $2 < k \leq |V(G)|$. For each edge uv assign the label r where r is the remainder when $f(u)$ is divided by $f(v)$ (or) $f(v)$ is divided by $f(u)$ according as $f(u) \geq f(v)$ or $f(v) \geq f(u)$. The function f is called a k-remainder cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$, $i, j \in \{1, \cdots, k\}$ where $v_f(x)$ denote the number of vertices labeled with x and $|\eta_e(0) - \eta_o(1)| \leq 1$ where $\eta_e(0)$ and $\eta_o(1)$ respectively denote the number of edges labeled with even integers and number of edges labeled with odd integers. A graph with admits a k-remainder cordial labeling is called a k-remainder cordial graph. In this paper we investigate the 3-remainder cordial labeling behavior of dumbbell graph, butterfly graph, umbrella graph, $C_3 \odot K_{1,n}$.

Key Words: Dumbbell graph, butterfly graph, umbrella graph, $C_3 \odot K_{1,n}$, Smarandache k-remainder cordial labeling.

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§1. Introduction

All graphs considered here are finite and simple. The origin of graph labeling is graceful labeling which was introduced by Rosa (1967). The concept of cordial labeling was introduced by Cahit [1]. Motivated by this Ponraj et al. [4, 6], introduced remainder cordial labeling of graphs and investigate the remainder cordial labeling behavior of several graphs. Also the notion of k-remainder cordial labeling introduced in [5] and investigate the $k-$remainder cordial labeling behavior of grid, subdivision of crown, subdivision of bistar, book, Jelly fish, subdivision of Jelly fish, mongolian tent, flower graph, sunflower graph and subdivision of ladder graph, $L_n \odot K_1$, $L_n \odot 2K_1$, $L_n \odot K_2$. Recently [9, 10] they investigate the 3-remainder cordial labeling behavior of the subdivision of the star, wheel, subdivision of the path, cycle, star, complete graph, comb, crown, wheel, subdivision of the comb, armed crown, fan, square of the path, $K_{1,n} \odot K_2$. In this paper we investigate the 3-remainder cordial labeling behavior of dumbbell graph, butterfly graph, umbrella graph, $C_3 \odot K_{1,n}$, etc. Terms are not defined here follows from Harary [3] and Gallian [2].

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§2. Preliminary Results

Definition 2.1 *The corona of G_1 with G_2, $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and p_1 copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2.*

Definition 2.2 *The graph obtained by joining two disjoint cycles, $u_1u_2\cdots u_nu_1$ and $v_1v_2\ldots v_nv_1$ with an edge u_1v_1 is called dumbbell graph Db_n.*

Definition 2.3 *The butterfly graph $BF_{m,n}$ is a two even cycles of the same order say C_n, sharing a common vertex with m pendant edges attached at the common vertex is called a butterfly graph.*

Definition 2.4 *The umbrella graph $U_{n,m}$ is obtained from a fan $F_n = P_n + K_1$ where $P_n : u_1, u_2, \cdots, u_n$ and $V(K_1) = \{u\}$ by pasting the end vertex of the path $P_m : v_1, v_2, \cdots, v_m$ to the vertex of K_1 of the fan F_n.*

§3. k-Remainder Cordial Labeling

Definition 3.1 *Let G be a (p, q) graph. Let f be a function from $V(G)$ to the set $\{1, 2, \cdots, k\}$ where k is an integer $2 < k \leq |V(G)|$. For each edge uv assign the label r where r is the remainder when $f(u)$ is divided by $f(v)$ (or) $f(v)$ is divided by $f(u)$ according as $f(u) \geq f(v)$ or $f(v) \geq f(u)$. The function f is called a k-remainder cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$, $i, j \in \{1, \cdots, k\}$, otherwise, Smarandachely if $|v_f(i) - v_f(j)| \geq 1$ or $|e_f(0) - e_f(1)| \geq 1$ for integers $i, j \in \{1, \cdots, k\}$, where $v_f(x)$ denote the number of vertices labeled with x and $|\eta_e(0) - \eta_o(1)| \leq 1$ where $\eta_e(0)$ and $\eta_o(1)$ respectively denote the number of edges labeled with even integers and number of edges labeled with odd integers. A graph with a k-remainder cordial labeling is called a k-remainder cordial graph.*

Now, we investigate the $3-$remainder cordial labeling behavior of the dumbbell graph Db_n.

Theorem 3.2 *The dumbbell graph Db_n is 3-remainder cordial for all n.*

Proof Let $C_n : u_1u_2\cdots u_nu_1$ and $C_n^{'} : v_1v_2\cdots v_nv_1$ be two disjoint cycles of the same order n. Let $V(Db_n) = V(C_n) \cup V(C_n^{'})$ and $E(Db_n) = E(C_n) \cup E(C_n^{'}) \cup \{u_1v_1\}$. Then the order and size of the dumbbell graph are $2n$ and $2n + 1$ respectively.

Case 1. $n \equiv 0 \pmod 3$.

Assign the labels $2, 3$ and 1 respectively to the vertices u_1, u_2 and u_3. Next assign the labels $1, 2$ and 3 to the vertices u_4, u_5 and u_6 respectively. Then assign the labels $2, 3$ and 1 respectively to the vertices u_7, u_8 and u_9. Then next assign the labels $1, 2$ and 3 to the vertices u_{10}, u_{11} and u_{12} respectively. Proceeding like this until we reach the vertex u_n. If n is odd then assign the labels $2, 3$ and 1 respectively to the vertices u_{n-2}, u_{n-1} and u_n. If n is even then assign the labels $1, 2$ and 3 respectively to the vertices u_{n-2}, u_{n-1} and u_n of C_n. On the other hand assign the labels $3, 2$ and 1 respectively to the vertices v_1, v_2 and v_3. Next assign the labels $1, 3$ and 2 to the vertices v_4, v_5 and v_6 respectively. Then assign the labels $3, 2$ and 1 respectively to the vertices v_7, v_8 and v_9. Then next assign the labels $1, 3$ and 2 to the vertices v_{10}, v_{11} and v_{12} respectively. Continuing like this until we reach the vertex v_n. If n is odd then assign the labels $3, 2$ and 1 respectively to the vertices v_{n-2}, v_{n-1} and v_n. If n is even then assign the labels $1, 3$ and 2 respectively to the vertices v_{n-2}, v_{n-1} and v_n of $C_n^{'}$. Table 1 shows that this vertex labeling is called 3-remainder cordial labeling of the dumbbell graph for $n \equiv 0 \pmod 3$.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	η_e	η_o
n is odd	$\frac{2n}{3}$	$\frac{2n}{3}$	$\frac{2n}{3}$	$n+1$	n
n is even	$\frac{2n}{3}$	$\frac{2n}{3}$	$\frac{2n}{3}$	n	$n+1$

Table 1

Case 2. $n \equiv 1 \pmod 3$.

Subcase 2.1 n is even.

Assign the labels to the vertices $u_i, 1 \le i \le n$ in the following way.

$$f(u_i) = \begin{cases} 2, & \text{if } i = 1, 3, 5, \cdots, i+2 \cdots, n-1, \\ 3, & \text{if } i = 2, 4, 6, \cdots, i+2 \cdots, n. \end{cases}$$

we consider the vertices $v_i, 1 \le i \le n$ of the cycle $C_n^{'}$. Assign the label 1 to the first $\frac{2n+1}{3}$ vertices $v_1, v_2, \cdots, v_{\frac{2n+1}{3}}$. Next assign the label 2 to the vertices $v_{\frac{2n+1}{3}+1}, v_{\frac{2n+1}{3}+2}, \cdots, u_{\frac{5n+4}{6}}$. Finally assign the label 3 to the remaining vertices of the cycle $C_n^{'}$.

Subcase 2.2 n is odd.

Assign the labels to the vertices $u_i, 1 \le i \le n$ in the following ways.

$$f(u_i) = \begin{cases} 3, & \text{if } i = 1, 3, 5, \cdots, i+2 \cdots, n, \\ 2, & \text{if } i = 2, 4, 6, \cdots, i+2 \cdots, n-1. \end{cases}$$

Next assign the labels to the vertices $v_i, 1 \le i \le n$ of the cycle $C_n^{'}$ in the following way. Assign the label 1 to the first $\frac{2n+1}{3}$ vertices $v_1, v_2, \ldots, v_{\frac{2n+1}{3}}$. Next assign the label 2 to the vertices $v_{\frac{2n+1}{3}+1}, v_{\frac{2n+1}{3}+2}, \ldots, u_{\frac{5n+1}{6}}$. Finally assign the label 3 to the remaining vertices of the cycle $C_n^{'}$. Table 2 shows that this vertex labeling is called 3-remainder cordial labeling of the dumbbell graph for $n \equiv 1 \pmod 3$.

Nature of n,$n \equiv 1 \pmod 3$	$v_f(1)$	$v_f(2)$	$v_f(3)$	η_e	η_o
n is odd	$\frac{2n+1}{3}$	$\frac{2n+1}{3}$	$\frac{2n-2}{3}$	$n+1$	n
n is even	$\frac{2n+1}{3}$	$\frac{2n+1}{3}$	$\frac{2n-2}{3}$	n	$n+1$

Table 2

Case 3. $n \equiv 2 \pmod 3$.

Fix the labels in the following pattern : $3, 2, 1, 1$ and 2 to the vertices u_1, u_2, u_3, u_{n-1} and u_n respectively and $2, 3, 2, 1$ and 3 to the vertices v_1, v_2, v_3, v_{n-1} and v_n respectively. Next assign the labels to the remaining vertices u_i, and $v_i, (4 \le i \le n-2)$ in the following two cases.

Subcase 3.1 First assign the labels to the vertices $u_i, 4 \le i \le n-2$. Assign the labels $1, 2$ and 3 to the vertices u_4, u_5 and u_6 respectively. Then assign the labels $2, 3$ and 1 respectively to the vertices u_7, u_8 and u_9. Then next assign the labels $1, 2$ and 3 to the vertices u_{10}, u_{11} and u_{12} respectively. Then assign the labels $2, 3$ and 1 respectively to the vertices u_{13}, u_{14} and u_{15}. Proceeding like this until we reach the vertex u_{n-2}. When n is odd then the vertices u_{n-4}, u_{n-3} and u_{n-2} are receive the labels $2, 3$ and 1 respectively. When n is even then the vertices u_{n-4}, u_{n-3} and u_{n-2} are receive the labels $1, 2$ and 3 respectively.

Subcase 3.2 We consider the vertices $v_i, (4 \leq i \leq n-2)$. Assign the labels to the vertices v_i for $(4 \leq i \leq n-2)$ as in subcase(i). Table 3 shows that this vertex labeling is called 3-remainder cordial labeling of the dumbbell graph for $n \equiv 2 \pmod 3$.

Nature of n,$n \equiv 2 \pmod 3$	$v_f(1)$	$v_f(2)$	$v_f(3)$	η_e	η_o
n is odd	$\frac{2n-1}{3}$	$\frac{2n+2}{3}$	$\frac{2n-1}{3}$	$n+1$	n
n is even	$\frac{2n-1}{3}$	$\frac{2n+2}{3}$	$\frac{2n-1}{3}$	n	$n+1$

Table 3

This completes the proof. □

Theorem 3.3 *The umbrella $U_{n,n}$ is 3-remainder cordial for all n.*

Proof Let $F_n = P_n + K_1$ where $P_n : u_1, u_2, \cdots, u_n$ and $V(K_1) = \{u\}$. Let $P_n' : v_1, v_2, \cdots, v_n$ be another path. Identify v_1 with u. Clearly the umbrella graph has $2n$ vertices and $3n-2$ edges.

Case 1. $n \equiv 0 \pmod 3$.

Subcase 1.1 n is odd.

Assign the labels to the vertices $u_i, (1 \leq i \leq n)$ as follows:

$$f(u_i) = \begin{cases} 2, & \text{if } i = 1, 3, 5, \ldots, i+2 \cdots, n, \\ 3, & \text{if } i = 2, 4, 6, \ldots, i+2 \cdots, n-1. \end{cases}$$

Next assign the labels to the vertices $v_i, 1 \leq i \leq n$. Assign the label 3 to the first $\frac{n+3}{6}$ vertices $v_1, v_2, \cdots, v_{\frac{n+3}{6}}$ and assign the label 1 consecutively to the vertices $v_{\frac{n+3}{6}+1}, v_{\frac{n+3}{6}+2}, \cdots, v_{\frac{5n+3}{6}}$. Next assign the label 2 to the remaining vertices.

Subcase 2. n is even.

Assign the labels to the vertices $u_i, (1 \leq i \leq n)$ as follows:

$$f(u_i) = \begin{cases} 2, & \text{if } i = 1, 3, 5, \cdots, i+2 \ldots, n-1, \\ 3, & \text{if } i = 2, 4, 6, \cdots, i+2 \ldots, n. \end{cases}$$

Next we consider the vertices $v_i, 1 \leq i \leq n$. Assign the label 3 to the first $\frac{n}{6}$ vertices $v_1, v_2, \cdots, v_{\frac{n}{6}}$ and assign the label 1 consecutively to the vertices $v_{\frac{n}{6}+1}, v_{\frac{n}{6}+2}, \cdots, v_{\frac{5n}{6}}$. Next assign the label 2 to the remaining vertices $v_{\frac{5n}{6}+1}, v_{\frac{5n}{6}+2}, \ldots, v_n$. Table 4 shows that this vertex labeling is called 3-remainder cordial labeling of $U_{n,n}$ for $n \equiv 0 \pmod 3$.

Nature of n,$n \equiv 0 \pmod 3$	$v_f(1)$	$v_f(2)$	$v_f(3)$	η_e	η_o
n is odd	$\frac{2n}{3}$	$\frac{2n}{3}$	$\frac{2n}{3}$	$\frac{3n-3}{2}$	$\frac{3n-1}{2}$
n is even	$\frac{2n}{3}$	$\frac{2n}{3}$	$\frac{2n}{3}$	$\frac{3n-2}{2}$	$\frac{3n-2}{2}$

Table 4

Case 2. $n \equiv 1 \pmod 3$.

Subcase 2.2 n is odd.

Assign the labels to the vertices $u_i, (1 \le i \le n)$ as follows:

$$f(u_i) = \begin{cases} 2, & \text{if } i = 1, 3, 5, \cdots, i+2 \cdots, n, \\ 3, & \text{if } i = 2, 4, 6, \cdots, i+2 \cdots, n-1. \end{cases}$$

Next assign the labels to the vertices $v_i, 1 \le i \le n$. Assign the label 3 to the first $\frac{n+5}{6}$ vertices $v_1, v_2, \cdots, v_{\frac{n+5}{6}}$ and assign the label 1 consecutively to the vertices $v_{\frac{n+5}{6}+1}, v_{\frac{n+5}{6}+2}, \cdots, v_{\frac{5n+7}{6}}$. Next assign the label 2 to the remaining vertices.

Subcase 2.2 n is even.

Assign the labels to the vertices $u_i, (1 \le i \le n)$ as follows:

$$f(u_i) = \begin{cases} 2, & \text{if } i = 1, 3, 5, \cdots, i+2 \cdots, n-1, \\ 3, & \text{if } i = 2, 4, 6, \cdots, i+2 \cdots, n. \end{cases}$$

Next we consider the vertices $v_i, 1 \le i \le n$. Assign the label 3 to the first $\frac{n+2}{6}$ vertices $v_1, v_2, \cdots, v_{\frac{n+2}{6}}$ and assign the label 1 to the vertices $v_{\frac{n+2}{6}+1}, v_{\frac{n+2}{6}+2}, \cdots, v_{\frac{5n+4}{6}}$ consecutively. Next assign the label 2 to the remaining vertices $v_{\frac{5n+4}{6}+1}, v_{\frac{5n+4}{6}+2}, \cdots, v_n$. Table 5 shows that this vertex labeling is called 3-remainder cordial labeling of $U_{n,n}$ for $n \equiv 1 \pmod 3$.

Nature of n,$n \equiv 0 \pmod 3$	$v_f(1)$	$v_f(2)$	$v_f(3)$	η_e	η_o
n is odd	$\frac{2n+1}{3}$	$\frac{2n-2}{3}$	$\frac{2n+1}{3}$	$\frac{3n-3}{2}$	$\frac{3n-1}{2}$
n is even	$\frac{2n+1}{3}$	$\frac{2n-2}{3}$	$\frac{2n+1}{3}$	$\frac{3n-2}{2}$	$\frac{3n-2}{2}$

Table 5

Case 3. $n \equiv 2 \pmod 3$.

Subcase 3.1 n is odd.

Assign the labels to the vertices $u_i, (1 \le i \le n)$ as follows:

$$f(u_i) = \begin{cases} 2, & \text{if } i = 1, 3, 5, \cdots, i+2 \cdots, n, \\ 3, & \text{if } i = 2, 4, 6, \cdots, i+2 \cdots, n-1. \end{cases}$$

Next assign the labels to the vertices $v_i, 1 \le i \le n$. Assign the label 3 to the first $\frac{n+1}{6}$ vertices $v_1, v_2, \cdots, v_{\frac{n+1}{6}}$ and assign the label 1 to the vertices $v_{\frac{n+1}{6}+1}, v_{\frac{n+1}{6}+2}, \cdots,$ $v_{\frac{5n+5}{6}}$ consecutively. Next assign the label 2 to the remaining $(\frac{n-5}{6})$ vertices.

Subcase 3.2 n is even.

Assign the labels to the vertices $u_i, (1 \le i \le n)$ as follows:

$$f(u_i) = \begin{cases} 2, & \text{if } i = 1, 3, 5, \cdots, i+2 \cdots, n-1, \\ 3, & \text{if } i = 2, 4, 6, \cdots, i+2 \cdots, n. \end{cases}$$

Next we consider the vertices $v_i, 1 \le i \le n$. Assign the label 3 to the first $\frac{n-2}{6}$ vertices

$v_1, v_2, \cdots, v_{\frac{n-2}{6}}$ and assign the label 1 to the vertices $v_{\frac{n-2}{6}+1}, v_{\frac{n-2}{6}+2}, \cdots, v_{\frac{5n+2}{6}}$ consecutively. Next assign the label 2 to the remaining $\frac{n-2}{6}$ vertices $v_{\frac{5n+2}{6}+1}, v_{\frac{5n+2}{6}+2}, \cdots, v_n$. Table 6 shows that this vertex labeling is called 3-remainder cordial labeling of $U_{n,n}$ for all $n \equiv 2 \pmod 3$.

Nature of n,$n \equiv 2 \pmod 3$	$v_f(1)$	$v_f(2)$	$v_f(3)$	η_e	η_o
n is odd	$\frac{2n+2}{3}$	$\frac{2n-1}{3}$	$\frac{2n-1}{3}$	$\frac{3n-3}{2}$	$\frac{3n-1}{2}$
n is even	$\frac{2n+2}{3}$	$\frac{2n-1}{3}$	$\frac{2n-1}{3}$	$\frac{3n-2}{2}$	$\frac{3n-2}{2}$

Table 6

This completes the proof. □

Theorem 3.4 *The butterfly graph $BF_{n,n}$ is 3-remainder cordial for all n.*

Proof Let $C_n : u_1u_2\cdots u_nu_1$ and $C_n' : v_1v_2\cdots v_nv_1$ be two cycles of the same order n. Identify the vertex u_1 with the vertex v_1. Let $w_1, w_2\cdots, w_n$ be the n-pendant vertices adjacent to the vertex u_1. Then the given graph has $3n-1$ vertices and $3n$ edges.

First assign the labels to the vertices $u_i, (1 \le i \le n)$ as follows:

$$f(u_i) = \begin{cases} 2, & \text{if } i = 1, 3, 5, \cdots, i+2\cdots, n-1, \\ 3, & \text{if } i = 2, 4, 6, \cdots, i+2\cdots, n. \end{cases}$$

Next assign the labels to the vertices $v_i, (2 \le i \le n)$. Assign the label 1 to the vertices $v_2, v_3, \ldots, v_n$. Finally assign the labels to the vertices $w_i, (1 \le i \le n)$ as follows:

$$f(w_i) = \begin{cases} 2, & \text{if } i = 1, 3, 5, \cdots, i+2\cdots, \frac{n}{2}, \\ 3, & \text{if } i = \frac{n}{2}+1, \frac{n}{2}+2, \cdots, i+2\cdots, n. \end{cases}$$

Thus $v_f(1) = n-1$, $v_f(2) = v_f(3) = n$ and $\eta_e = \frac{3n}{2} = \eta_o$. Hence this vertex labeling is called 3-remainder cordial labeling of butterfly graph for all n. □

Theorem 3.5 *The graph $C_3 \odot K_{1,n}$ is 3-remainder cordial for all n.*

Proof Let $V(C_3 \odot K_{1,n}) = \{u, v, w, u_i, v_i, w_i : 1 \le i \le n\}$, $E(C_3 \odot K_{1,n}) = \{uv, vw, wu, uu_i, vv_i, ww_i : 1 \le i \le n\}$. Clearly the order and size of the given graph are $3n+3$ and $3n+3$ respectively.

Fix Tables 1, 2 and 3 respectively to the central vertices u, v and w of $C_3 \odot K_{1,n}$ and also fix the label 3 to the vertices $v_1, v_2, v_3, \cdots, v_n$ into the following two cases.

Case 1. n is even.

First we consider the vertices $u_i, (1 \le i \le n)$. Assign the label 1 consecutively to the vertices $u_1, u_2, \cdots, u_{\frac{n+1}{2}}$. Next assign the label 2 to the remaining vertices $u_{\frac{n+1}{2}+1}, u_{\frac{n+1}{2}+2}, \cdots, u_n$.

Next we consider the vertices $w_i, (1 \le i \le n)$. Assign the label 2 consecutively to the vertices $w_1, w_2, \cdots, w_{\frac{n+1}{2}}$. Next assign the label 1 to the remaining vertices $w_{\frac{n+1}{2}+1}, w_{\frac{n+1}{2}+2}, \cdots, w_n$.

Case 2. n is odd.

Assign the label 1 to the first $(\frac{n}{2})$ vertices $u_1, u_2, \cdots, u_{\frac{n}{2}}$ and assign the label 2 to the remaining $(\frac{n}{2})$ vertices $u_{\frac{n}{2}+1}, u_{\frac{n}{2}+2}, \ldots, u_n$. Next we consider the vertices $w_i, (1 \le i \le n)$. Assign the label 2 consecutively to the vertices $w_1, w_2, \cdots, w_{\frac{n}{2}}$ and assign the label 1 to the next $(\frac{n}{2})$ vertices

$w_{\frac{n}{2}+1}, w_{\frac{n}{2}+2}, \cdots, w_n$. Table 7 shows that this vertex labeling is called 3-remainder cordial labeling of $C_3 \odot K_{1,n}$ for all n.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	η_e	η_o
n is odd	$n+1$	$n+1$	$n+1$	$\frac{3n+3}{2}$	$\frac{3n+3}{2}$
n is even	$n+1$	$n+1$	$n+1$	$\frac{3n+4}{2}$	$\frac{3n+2}{2}$

Table 7

This completes the proof. □

Example 3.6 A 3-remainder cordial labeling of $C_3 \odot K_{1,9}$ is shown in Figure 1.

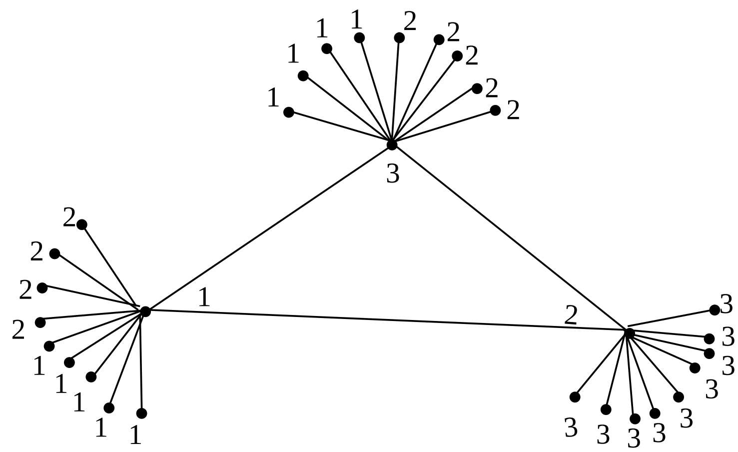

Figure 1

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Famous Words

Could a special solutions of Einstein's gravitational equations be applied to the whole universe? The answer is obviously negative! However, the Schwarzschild spacetime is its a special solution in an assumption that all matters are spherically symmetric distributed in the universe of vacuum, which results the Big Bang hypothesis and the standard model on universe. So, we are applying a special solution for the universe and believe it without a shadow of doubt in any place of the universe. Why it happened because we are all fond of the symmetry, the uniformity on space and we are firmly believing the spacetime structure of the universe should be so by observed datum of humans, at least in the nearby airspace of the earth. But, it is only an understanding of humans ourself on the unverse, partially or locally. (Extracted from the paper: Science's Dilemma - a Review on Science with Applications, *Progress in Physics*, Vol.15 (2019), 78-85.)

By Dr.Linfan MAO, a Chinese mathematician, philosophical critic.

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[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.
[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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