Few Results on Paired Domination Number of Some Graphs

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***Abstract*— The paired dominating set D of graph G is the subset of vertex set of graph G such that the subset D is dominating set and the subgraph induced by D contains perfect Matching. The paired domination number is the minimum cardinality of a paired dominating set D of graph G. In this paper we discuss the paired domination number of middle graph and total graph of paths and cycles. It will be followed by the discussion on paired domination number of splitting graph, shadow graph and the graphs obtained by switching of the vertices of paths and cycles.**

***Keywords*— Dominating set, Paired- Dominating set, Paired- Domination Number, Middle Graph, Total Graph, Splitting Graph, Shadow Graph, Switching of vertex.**

1. Introduction

Throughout the paper we consider finite, undirected and simple graph G. For the graph theoretical concepts, we refer Clark and Holton [9]. The concept of dominating set and domination number is introduced by Ore [8] in 1962 and Berge [7] in 1958. The literature of this topic has been surveyed in the books of Haynes *et al*. [10, 11]. The present work focuses on one of the variants of domination named paired-domination. The paired-domination was introduced by Haynes and Slater [1] in 1998. A paired-dominating set of a graph G with no isolated vertex is a dominating set of vertices whose induced subgraph has a perfect matching. The paired-domination number of G, denoted by $\gamma_{pr}(G)$, is the minimum cardinality of a paired domination set of G.

Many researchers have studied on paired-domination. Fitzpatrick and Hartnell [2] focused on the graphs which have a maximal matching whose end vertices form a minimum paired-dominating set. Chellai *et al.* [3] had given sharp upper bounds on paired-domination number of trees that improved bounds for some cases. Henning [4] showed that there are exactly ten graphs that achieve equality in the bound $\gamma_{pr}(G) \leq \frac{2n}{3}$. The author proved, for $n \geq 14$, $\gamma_{pr}(G) \leq \frac{2(n-1)}{3}$ and characterized the graphs that achieve equality in this bound while Favaron *et al.* [5] studied the bound on the sum $\gamma_{pr}(G) + \gamma_{pr}(\bar{G})$. Isaac and Pandya [6] investigated paired-domination number of k^{th} power graph of path and cycle, and degree splitting of path.

2. Main Results

In this section we study the paired domination number of middle graph and total graph of path P_n and cycle C_n. It will be followed by the discussion on paired domination number of splitting graph, shadow graph and the graphs obtained by switching of the vertices.

Definition 2.1. The middle graph $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident on it.

Theorem 2.2. $\gamma_{pr}\left(M(P_n)\right) = 2\left\lceil\frac{n}{3}\right\rceil$.

Proof. Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$ be the vertices and $E(P_n) = \{e_1, e_2, \dots, e_{n-1}\}$ be the edge set of P_n. By the definition of middle graph, $V\left(M(P_n)\right) = V(P_n) \cup E(P_n)$.

In $M(P_n)$, the set of vertices $\{e_1, e_2\}$ and $\{e_{n-2}, e_{n-1}\}$ dominates six vertices while $\{e_i, e_{i+1}\}$ dominates seven vertices including themselves where $i = 2, 3, \dots, n-2$.

When $n \equiv 0 (\text{mod } 3)$, and if we consider the set $D = \{e_1, e_2, e_4, e_5, e_7, e_8, \dots, e_{n-2}, e_{n-1}\}$we observe that D is a paired-dominating set with cardinality $2\left(\frac{n}{3}\right)$ and by removal of any pair from D, it will not be a dominating set. Hence, $\gamma_{pr}\left(M(P_n)\right) = 2\left(\frac{n}{3}\right)$ if $n \equiv 0 \ (\text{mod } 3)$.

When $n \not\equiv 0 \ (\text{mod } 3)$ we get $D_1 = D \cup \{e_{n-1}, v_n\}$ as a paired dominating set using the pair of vertices $\{e_{n-1}, v_n\}$ and hence, $\gamma_{pr}\left(M(P_n)\right) = 2\left(\frac{n}{3}\right) + 2 = 2\left\lceil\frac{n}{3}\right\rceil$.

Thus, in general, $\gamma_{pr}\left(M(P_n)\right) = 2\left\lceil\frac{n}{3}\right\rceil$.

Theorem 2.3. $\gamma_{pr}\left(M(C_n)\right) = 2\left\lceil\frac{n}{3}\right\rceil$.

Proof. Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$be the vertex set and $E(C_n) = \{e_1, e_2, \dots, e_n\}$ be the edge set of cycle C_n. By the definition of middle graph, $V\left(M(C_n)\right) = V(C_n) \cup E(C_n)$. In $M(C_n)$, the set of vertices $\{e_i, e_{i+1}\}$as well as $\{e_1, e_n\}$ dominates seven vertices including themselves; $1 \le i \le n-1$.

When $n \equiv 0 (\text{mod } 3)$, the set $D = \{e_1, e_2, e_4, e_5, e_7, e_8, \dots, e_{n-2}, e_{n-1}\}$ is a paired-dominating set with cardinality $2\left(\frac{n}{3}\right)$ and by removal of any pair from the set D, it will not be a dominating set.

Hence, $\gamma_{pr}\left(M(C_n)\right) = 2\left(\frac{n}{3}\right)$ if $n \equiv 0 \ (\text{mod } 3)$.

If $n \not\equiv 0 (\text{mod } 3)$, we need to include a pair of vertices $\{e_n, v_n\}$ to the paired-dominating set D and hence, $\gamma_{pr}\left(M(C_n)\right) = 2\left(\frac{n}{3}\right) + 2 = 2\left\lceil\frac{n}{3}\right\rceil$.

Thus, in general, $\gamma_{pr}\left(M(C_n)\right) = 2\left\lceil\frac{n}{3}\right\rceil$.

Definition 2.4. The total graph of a graph $T(G)$ of G, is the graph with the vertex set $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G.

Theorem 2.5. $\gamma_{pr}\left(T(P_n)\right) = 2\left\lceil\frac{n-1}{3}\right\rceil$.

Proof. Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$ be the vertices and $E(P_n) = \{e_1, e_2, \dots, e_{n-1}\}$ be the edge set of P_n. By the definition of total graph, $V(T(P_n)) = V(P_n) \cup E(P_n)$. The set of vertices $\{e_i, v_i\}$ where $i = 2,3, \dots, n-1$ dominates 6 vertices in $T(P_n)$ and thus, we have the following cases:

Case (1): $n \equiv 0 \pmod 3$

Consider the set $D = \{v_2, e_2, v_5, e_5, v_8, e_8, \dots, v_{n-1}, e_{n-1}\}$. Clearly D is a paired-dominating set with cardinality $2\left\lceil\frac{n-1}{3}\right\rceil$. The set D is the minimum paired-dominating set of $T(P_n)$ as by removal of any pair from the set, it will not be a dominating set. Hence, $\gamma_{pr}(T(P_n)) = 2\left\lceil\frac{n-1}{3}\right\rceil$ when $n \equiv 0 (\text{mod } 3)$.

Case (2): $n \equiv 1 \pmod 3$

Consider the set $D = \{v_2, e_2, v_5, e_5, v_8, e_8, \dots, v_{n-5}, e_{n-5}, v_{n-2}, v_{n-1}\}$. Clearly D is a paired-dominating set with cardinality $2\left(\frac{n-1}{3}\right)$. The set D is the minimum paired-dominating set of $T(P_n)$ as by removal of any pair from the set, it will not be a dominating set. Hence, $\gamma_{pr}(T(P_n)) = 2\left(\frac{n-1}{3}\right)$when $n \equiv 1 \pmod 3$.

Case (3): $n \equiv 2 \pmod 3$

The set $D = \{v_2, e_2, v_5, e_5, v_8, e_8, \cdots, v_{n-2}, e_{n-2}, v_n, e_{n-1}\}$, is a paired-dominating set with cardinality$2\left(\frac{n+1}{3}\right)$. The set D is the minimum paired-dominating set of $T(P_n)$ as by removal of any pair from the set, it will not be a dominating set. Hence, $\gamma_{pr}(T(P_n)) = 2\left(\frac{n+1}{3}\right)$. As $n \equiv 2 (\text{mod } 3)$, $2\left(\frac{n+1}{3}\right) = 2\left\lceil\frac{n-1}{3}\right\rceil$.

Thus, in general, $\gamma_{pr}(T(P_n)) = 2\left\lceil\frac{n-1}{3}\right\rceil$.

Corollary 2.6. $\gamma_{pr}(T(C_n)) = 2\left\lceil\frac{n}{3}\right\rceil$.

Proof. The proof is analogous to the proof of Theorem 2.3 as minimum paired-dominating set of $T(C_n)$ is same as the minimum paired-dominating set of $M(C_n)$. Hence$\gamma_{pr}(T(C_n)) = \gamma_{pr}(M(C_n)) = 2\left\lceil\frac{n}{3}\right\rceil$.

Definition2.7. For a graph $G = (V, E)$ the splitting graph $S'(G)$is obtained by taking vertex v' for each vertex $v \in G$ and join each v' to all vertices of G adjacent to v.

Theorem 2.8. $\gamma_{pr}(S'(G)) = \gamma_{pr}(G)$.

Proof. The minimum paired-dominating set of $S(G)$ is same as the minimum paired-dominating set of any graph$G$by the definition of splitting graph and hence,$\gamma_{pr}(S'(G)) = \gamma_{pr}(G)$.

Definition 2.9. The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G, say G' & $G"$. Join each vertex u' in G' to the neighbors of the corresponding vertex $u"$ in $G"$.

Theorem 2.10. $\gamma_{pr}(D_2(G)) = \gamma_{pr}(G)$.

Proof. The minimum paired-dominating set of $D_2(G)$ is same as the minimum paired-dominating set of $G$by the definition of shadow graph of a graph and hence,$\gamma_{pr}\left(D_2(G)\right) = \gamma_{pr}(G)$.

Definition 2.11. Vertex switching G_v of a graph G is the graph obtained by taking a vertex v of G, removing all the edges to v and adding edges joining v to every other vertex which are not adjacent to v in G.

Theorem 2.12. *For* $n \geq 4, \gamma_{pr}\left(C_{n_v}\right) = \begin{cases} 2, & 4 \leq n \leq 5. \\ 4, & n > 5. \end{cases}$

Proof.

Case (1): As $C_{4_v} \cong K_{1,3}$ and as $\gamma_{pr}\left(K_{1,3}\right) = 2$, $\gamma_{pr}(C_{4_v}) = 2$.

Case (2): $n = 5$

By switching the vertex v and let $D = V - N[v]$,D is minimum paired-dominating set.

Clearly, $|D| = 5 - 3 = 2$ and hence, $\gamma_{pr}(C_{5_v}) = 2$.

Case (3): $n > 5$

In the graph obtained by switching the vertex v of $C_n$we observe that$v$ is adjacent to every vertex ofC_{n_v}, except two pendant vertices, say v_1 and v_2. So, the vertex v will dominate all vertices exceptv_1 and v_2. In order to dominate them, we need to consider two vertices which are adjacent to v_1 and v_2 and hence, those vertices along with the vertex v form a minimum dominating set. To make this set a minimum paired-dominating set, we need to consider any vertex, not from the above set, and hence the minimum paired-dominating set will contain 4 vertices and therefore, $\gamma_{pr}\left(C_{n_v}\right) = 4$.

Observation 2.13. *For path P_n, we are not switching the vertices v which are adjacent to pendant vertices, as the resultant graph P_{n_v}will be disconnected with a component having only one vertex and we cannot obtain the paired-domination number of P_{n_v}.*

Theorem 2.14. *For* $n \geq 4$,

$$\gamma_{pr}\left(P_{n_v}\right) = \begin{cases} 2, & \text{if } v \text{ is a pendant vertex.} \\ 4, & \text{if } v \text{ is any internal vertex not adjacent to any pendant vertex} \end{cases}$$

Proof.

Case (1): v is a pendant vertex.

By switching a pendant vertex v in P_n, v will be adjacent to all vertices except the vertex adjacent to it, say u, in the graph P_n. That is, the vertex v will dominate all the vertices except u in P_{n_v}. So, the minimum paired-dominating set will be of cardinality 2. Hence, $\gamma_{pr}(P_{n_v}) = 2$.

Case (2): v is an internal vertex not adjacent to any pendant vertex.

When we switch the internal vertex, it will dominate all vertices except the two vertices which are adjacent to it. So the minimum paired-dominating set will be of cardinality 4. Hence, $\gamma_{pr}(P_{n_v}) = 4$.

3. Conclusions

In this paper we are trying to discuss the pair domination number of the standard graphs, path and cycle, obtained by some graph operations such as middle graph, total graph, splitting graph, shadow graph and the graphs obtained after switching of vertices.

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