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ON SOLVING VARIATIONAL INEQUALITY FOR SM-ITERATION

SAVITA RATHEE, MONIKA SWAMI*

Department of Mathematics, Maharshi Dayanand University, Rohtak, 124001, India

Abstract. The fundamental motivation behind this paper is to solve variational inequality for SM-iterative method under some gentle conditions. We focus on calculating the common results of variational inequality and of SM-iteration. As application part, convex minimization problem is solved under modified SM-algorithm. Numerical example is supplied to validate our main result. Our result holds comparison between the SM- and S-algorithms.

Keywords: variational inequality; fixed point; convex minimization problem.

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1. INTRODUCTION

Consider $\mathfrak{C}$ be "closed and convex" non empty subset of "Hilbert space $\mathscr{H}$" with innerproduct space $\langle , \rangle$ and induced norm $\|.\|$. Regard $\mathfrak{T}$ be self mapping over $\mathfrak{C}$ with projection $P_{\mathfrak{C}}$ of $\mathscr{H}$ onto the convex set $\mathfrak{C}$. Review some definitions over nonlinear operator $\mathfrak{T} : \mathfrak{C} \subset \mathscr{H} \to \mathscr{H}$ is said to be:

(1) "μ-Lipschitzian if for all $a, b \in \mathfrak{C}$, there exists a constant $\mu > 0$ such that

$$\|\mathfrak{T}a - \mathfrak{T}b\| \leq \mu \|a - b\|$$

*Corresponding author
E-mail address: monikaswami06@gmail.com
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(2) non expansive if for all $a, b \in \mathfrak{C}$, we have

$$\|\mathfrak{T}a - \mathfrak{T}b\| \leq \|a - b\|$$

(3) ([11]) relaxed (ν, r)-cocoercive if for all $a, b \in \mathfrak{C}$, there exists constants $\nu, r > 0$ such that

$$\langle \mathfrak{T}a - \mathfrak{T}b, a - b \rangle \geq -\nu \|\mathfrak{T}a - \mathfrak{T}b\|^2 + r\|a - b\|^2\text{''}.$$

Lemma 1. *(*[4]*) "Let $P_{\mathfrak{C}} : \mathscr{H} \to \mathfrak{C}$ be a projection mapping. Then $P_{\mathfrak{C}}$ is nonexpansive, that is, $\|P_{\mathfrak{C}}a - P_{\mathfrak{C}}b\| \leq \|a - b\|$ for all $a, b \in \mathscr{H}$".*

In 1964, Stampacchia ([10]) introduced the "variational inequality problem" as given:

$$\text{finding } u \in \mathscr{H} \text{ such that } \langle \mathfrak{T}a, b - a \rangle \geq 0 \ \text{ for all } b \in \mathfrak{C}.$$

The proplem is represented by $VI(\mathfrak{C}, \mathfrak{T})$ and solution set is indicated by $\Omega(\mathfrak{C}, \mathfrak{T}) = \{a \in \mathfrak{C} : \langle \mathfrak{T}a, b - a \rangle \geq 0, \forall\, b \in \mathfrak{C}\}$.
The following lemma gives us the equality connection between "variational inequality" and "fixed point problem".

Lemma 2. *(*[4]*) "Consider $P_{\mathfrak{C}} : \mathfrak{C} \to \mathscr{H}$ be a projection mapping. For a given $c \in \mathscr{H}$,*

$$a \in \mathfrak{C} \text{ satisfies } \langle a - c, b - a \rangle \geq 0, \text{ for all } b \in \mathfrak{C} \text{ if and only if } a = P_{\mathfrak{C}}[c].$$

Moreover,

$$a \in \Omega(\mathfrak{C}, \mathfrak{T}) \text{ if and only if } u = P_{\mathfrak{C}}[a - \sigma \mathfrak{T}a],$$

where $\sigma > 0$ is a constant [7]."

We deal with S as a "nonexpansive mapping" and $\mathfrak{F}(S)$ represent the solution set of "fixed point" of mapping S. If $a^* \in \mathfrak{F}(S) \cap \Omega(\mathfrak{C}, \mathfrak{T})$, then $a^* \in \mathfrak{F}(S)$ and $a^* \in \Omega(\mathfrak{C}, \mathfrak{T})$. From lemma (2), We have discovered that

$$a^* = Sa^* = P_{\mathfrak{C}}[a^* - \sigma \mathfrak{T}a^*] = SP_{\mathfrak{C}}[a^* - \sigma \mathfrak{T}a^*], \tag{1}$$

where $\sigma > 0$ is a constant ([7]). In 2007, Noor [7] use the fixed formulation techique to solve the iterative method defined as follows:

$$
\begin{aligned}
a_0 &\in \mathfrak{C},\\
a_{n+1} &= (1-\beta_n^{(1)})a_n + \beta_n^{(1)} SP_{\mathfrak{C}}[b_n - \sigma\mathfrak{T}b_n],\\
b_n &= (1-\beta_n^{(2)})a_n + \beta_n^{(2)} SP_{\mathfrak{C}}[c_n - \sigma\mathfrak{T}c_n],\\
c_n &= (1-\beta_n^{(3)})a_n + \beta_n^{(3)} SP_{\mathfrak{C}}[a_n - \sigma\mathfrak{T}a_n],
\end{aligned}
\tag{2}
$$

where $\beta_n^{(k)}$ are real sequences, for all $k = 1,2,3$ in $[0,1]$ and S is "nonexpansive operator" ([7]). From algorithm (2), there are several algorithm as special cases for solving "variational inequality" with "nonexpansive mappings".

In [7], Noor approximate the algorithm (2) to a point of $\mathfrak{F}(S)\cap\Omega(\mathfrak{C},\mathfrak{T})$, which is common solution of fixed point set of nonexpansive mapping and "variational inequalities". Noor [7] proved the following theorem.

Theorem 3. *"Let $\mathfrak{C}$ be a closed convex subset of a real Hilbert space $\mathscr{H}$. Let $\mathfrak{T}$ be a relaxed (ν,r)-cocoercive and μ-Lipschitzian mapping of $\mathfrak{C}$ into $\mathscr{H}$, and S be a nonexpansive mapping of $\mathfrak{C}$ into $\mathfrak{C}$ such that $\mathfrak{F}(S)\cap\Omega(\mathfrak{C},\mathfrak{T}) \neq \phi$. Let $\{a_n\}$ be a sequence defined by algorithm* (2)*, for any initial point $a_0 \in \mathfrak{C}$, with conditions*

$$
0 < \sigma < 2(r-\nu\mu^2)/\mu^2, \nu\mu^2 < r, \tag{3}
$$

$\beta_n^{(k)} \in [0,1]$ for all $k = 1,2,3$ and for all $n \in N$ and $\sum_{n=0}^{\infty}\beta_n^{(1)} = \infty$, then $\{a_n\}$ obtained from algorithm (2) *converges strongly to $a^* \in \mathfrak{F}(S)\cap\Omega(\mathfrak{C},\mathfrak{T})$".*

In 2007, Agarwal [1] gives S-algorithm which is better than "Picard's iteration" [8] and "Mann iteration" [6] under contraction mappings. Construction of S-algorithm is independent of Picard's and Mann iterations, which is the following:

$$
\begin{aligned}
a_0 &\in \mathfrak{C},\\
s_{n+1} &= (1-\alpha_n^{(1)})\mathfrak{T}(s_n) + \alpha_n^{(1)}\mathfrak{T}(u_n),\\
u_n &= (1-\alpha_n^{(2)})s_n + \alpha_n^{(2)}\mathfrak{T}(s_n), \text{ for all } n \in N.
\end{aligned}
$$

in which $\{\alpha_n^{(k)}\}$, for $k = 1,2$ are real sequences in [0,1] with $\mathfrak{T}$ as a self-mapping over $\mathfrak{C}$. In 2021, Erturk [3] used the same formulation tehnique for S-algorithm to create a reasonable and common solution to the "fixed point and variational problems", which are described as follows:

$$\begin{aligned} & a_0 \in \mathfrak{C}, \\ (4) \qquad & s_{n+1} = (1-\alpha_n^{(1)})\Xi(s_n) + \alpha_n^{(1)}\Xi(u_n), \\ & u_n = (1-\alpha_n^{(2)})s_n + \alpha_n^{(2)}\Xi(s_n), \text{ for all } n \in N. \end{aligned}$$

where $\Xi : \mathfrak{C} \to \mathfrak{C}$ is an operator illustrated as

$$(5) \qquad \text{"}\Xi = SP_{\mathfrak{C}}[I - \sigma\mathfrak{T}].\text{"}$$

Here $S : \mathfrak{C} \to \mathfrak{C}, \mathfrak{T} : \mathfrak{C} \to \mathscr{H}$ and $P_{\mathfrak{C}} : \mathscr{H} \to \mathfrak{C}$ defined in theorem (3) with the imposed condition (3), then for all $a, b \in \mathfrak{C}$, we obtain

$$\begin{aligned} \|\Xi(a) - \Xi(b)\| =& \|SP_{\mathfrak{C}}[a - \sigma\mathfrak{T}a] - SP_{\mathfrak{C}}[b - \sigma\mathfrak{T}b]\| \\ \le& \|P_{\mathfrak{C}}[a - \sigma\mathfrak{T}a] - P_{\mathfrak{C}}[b - \sigma\mathfrak{T}b]\| \\ \le& \|a - \sigma\mathfrak{T}a - (b - \sigma\mathfrak{T}b)\| \\ (6) \qquad =& \|a - b - \sigma(\mathfrak{T}a - \mathfrak{T}b)\| \end{aligned}$$

Now, from the "relaxed (ν, r)-cocoercive"and "μ-Lipschitzian"definition on $\mathfrak{T}$, we get

$$\begin{aligned} \|a - b - \sigma(\mathfrak{T}a - \mathfrak{T}b)\|^2 =& \|a-b\|^2 - 2\sigma\langle \mathfrak{T}a - \mathfrak{T}b, a - b\rangle + \sigma^2\|\mathfrak{T}a - \mathfrak{T}b\|^2 \\ \le& \|a-b\|^2 + 2\sigma\nu\|\mathfrak{T}a - \mathfrak{T}b\|^2 - 2\sigma r\|a-b\|^2 + \sigma^2\|\mathfrak{T}a - \mathfrak{T}b\|^2 \\ (7) \qquad \le& [1 + 2\sigma\nu\mu^2 - 2\sigma r + \sigma^2\mu^2]\|a-b\|^2 = \tau^2\|a-b\|^2, \end{aligned}$$

which implies

$$(8) \qquad \|a - b - \sigma(\mathfrak{T}a - \mathfrak{T}b)\| \le \tau\|a-b\|,$$

where

$$(9) \qquad \tau = \sqrt{1 + 2\sigma\nu\mu^2 - 2\sigma r + \sigma^2\mu^2}.$$

Substituting (8) in (6), we get

$$\|\Xi(a)-\Xi(b)\| \leq \tau\|a-b\|. \tag{10}$$

Condition (3) implies $\tau < 1$ which gives us that $\Xi : \mathfrak{C} \to \mathfrak{C}$ is contration. Thus, by "Banach Contraction Principle", Ξ has a unique "fixed point" and the assumption $\mathfrak{F}(S) \cap \Omega(\mathfrak{C},\mathfrak{T}) \neq \phi$ in theorem (3) is important.

Definition 1.1. ([5]) "Let $\{a_n^{(i)}\}_{n=0}^{\infty}$ for $i = 1,2$ be two iterations converging to the same fixed point a^*. We say that $\{a_n^{(1)}\}_{n=0}^{\infty}$ converges faster than $\{a_n^{(2)}\}_{n=0}^{\infty}$ to a^* if

$$\lim_{n\to\infty} \frac{\|a_n^{(1)}-a^*\|}{(\|a_n^{(2)}-a^*\|)} = 0."$$

In 2020, Rathee [9] introduced the SM-iteration, defined as follow:

$$\begin{aligned}
a_0 &\in \mathfrak{C} \\
a_{n+1} &= S((1-\alpha_n^{(3)})Sc_n + \alpha_n^{(3)}Sb_n) \\
b_n &= S((1-\alpha_n^{(4)})a_n + \alpha_n^{(4)}c_n) \\
c_n &= Sa_n
\end{aligned}$$

where $\{\alpha_n^{(3)}\}$ and $\{\alpha_n^{(4)}\}$ in [0,1].

The primary goal of this paper is to suggest the modified SM-algorithm for solving $VI(\mathfrak{C},S,\mathfrak{T})$ as under

$$\begin{aligned}
a_{n+1} &= S((1-\alpha_n^{(3)})\Xi c_n + \alpha_n^{(3)}\Xi b_n) \\
b_n &= S((1-\alpha_n^{(4)})a_n + \alpha_n^{(4)}c_n) \\
c_n &= \Xi a_n
\end{aligned} \tag{11}$$

where Ξ is explained by (6).

We prove that iteration (11) strongly converges to the solution of $VI(\mathfrak{C},S,\mathfrak{T})$ and compare the convergence of algorithm (11) with algorithm (4). Also we depict a numerical example to support our result. On the other hand, an altered version of algorithm (11) is presented for solving "convex minimization problems" with the support of numerical example.

The accompanying lemmas will be expected to understand our results:

Lemma 4. [2] *"Let* $\{\Upsilon_n^{(i)}\}_{n=0}^{\infty}$ *for* $i=1,2$ *be a non-negative sequences of real numbers satisfying*

$$\Upsilon_{n+1}^{(1)} \leq \lambda \Upsilon_n^{(1)} + \Upsilon_n^{(2)} \text{ for all } n \in N,$$

where $\lambda \in [0,1)$ *and* $\lim_{n\to\infty} \Upsilon_n^{(2)} = 0$. *Then,* $\lim_{n\to\infty} \Upsilon_n^{(1)} = 0$.*"*

2. Main Results

Theorem 5. *Assume* $\mathscr{H}, \mathfrak{C}, S$ *and* $\mathfrak{T}$ *be defined in theorem* (3) *and* Ξ *in* (5) *with imposed condition in* (9) *gratified. Consider* $\{a_n\}$ *be an iteration given by algorithm* (11) *with the sequences* $\{\alpha_n^{(3)}\}$ *and* $\{\alpha_n^{(4)}\}$ *in [0,1], then* $\{a_n\}$ *strongly convergent to* $a^* \in \mathfrak{F}(S) \cap \Omega(\mathfrak{C}, S, \mathfrak{T})$ *with the appropriate condition:*

$$\|a_{n+1} - a^*\| \leq \tau^{n+1} \|a_0 - a^*\| \text{ for all } n \in N,$$

in which τ *is expressed in* (9).

Proof. The contractivness condition of operator Ξ gives the existence of solution $a^* \in \mathfrak{F}(S) \cap \Omega(\mathfrak{C}, S, \mathfrak{T})$ of $VI(\mathfrak{C}, S, \mathfrak{T})$ which is unique in number. As we have

$$a^* = S((1-\alpha_n^{(3)})\Xi a^* + \alpha_n^{(3)} \Xi a^*) = S((1-\alpha_n^{(4)})a^* + \alpha_n^{(4)} a^*) = \Xi a^*. \tag{12}$$

We pursue from (10), (11) and (12), that

$$\begin{aligned}
\|a_{n+1} - a^*\| &= (1-\alpha_n^{(3)})\tau\|c_n - a^*\| + \alpha_n^{(3)}\tau\|b_n - a^*\| \\
&\leq (1-\alpha_n^{(3)})\tau^2\|a_n - a^*\| + \alpha_n^{(3)}\tau\{(1-\alpha_n^{(4)})\|a_n - a^*\| + \alpha_n^{(4)}\tau\|a_n - a^*\|\} \\
&= [(1-\alpha_n^{(3)})\tau^2 + \alpha_n^{(3)}\tau(1-\alpha_n^{(4)}(1-\tau))]\|a_n - a^*\| \\
&= \tau[(1-\alpha_n^{(3)})\tau + \alpha_n^{(3)}(1-\alpha_n^{(4)}(1-\tau))]\|a_n - a^*\| \\
&\leq \cdots \\
&= \tau^{n+1}\prod_{i=0}^{n}[(1-\alpha_i^{(3)})\tau + \alpha_i^{(3)}(1-\alpha_i^{(4)}(1-\tau))]\|a_0 - a^*\| \\
&= \tau^{n+1}\prod_{i=0}^{n}[\tau + \alpha_i^{(3)}(1-\tau) - \alpha_i^{(3)}\alpha_i^{(4)}(1-\tau)]\|a_0 - a^*\| \qquad (13)
\end{aligned}$$

where τ is defined by (9).

Since $\{\alpha_n^{(3)}\}, \{\alpha_n^{(4)}\} \subset [0,1]$ and $\tau < 1$, then $\tau + \alpha_i^{(3)}(1-\tau) - \alpha_i^{(3)}\alpha_i^{(4)}(1-\tau) < 1$ for every $i = 0,1,2,...n.$. As a consequence of this, (13) gives

$$\|a_{n+1} - a^*\| \leq \tau_{n+1}\|a_0 - a^*\| \text{ for all } n \in N. \tag{14}$$

By taking limit on both side in (14), there is a $\lim_{n\to\infty}\|a_{n+1} - a^*\| = 0$ which gives us $a_n \to a^*$ as $n \to \infty$. □

Theorem 6. *Assume $\mathscr{H}, \mathfrak{C}, S, \mathfrak{T}$ and Ξ are defined in theorem* (5) *and τ expressed in equation* (9). *Consider $\{s_n\}_{n=0}^{\infty}$ with $\{a_n\}_{n=0}^{\infty}$ be two iterations explained in equations* (4) *and* (11), *jointly, where $\{\alpha_n^{(k)}\}_{n=0}^{\infty} \subset [0,1]$ for all $k = 1,2,3,4$. Assume that the facts in* (3) *holds. Then the respective assertions hold:*

(i) If

$$\left\{\frac{\max_{\{k=1,2,3,4\}}\{1-\alpha_n^{(k)}(1-\tau)\}}{a_n^{(1)}}\right\}_{n=0}^{\infty}$$

is "bounded" with $\sum_{n=0}^{\infty}\alpha_n^{(1)} = \infty$, then $\{a_n - s_n\}_{n=0}^{\infty}$ strongly convergent to point 0 with the accompanying calculation:

$$\|a_{n+1} - s_{n+1}\| \leq \tau\|a_n - s_n\| + (\tau^2 + 3\tau)\max_{\{k=1,2,3,4\}}\{1-\alpha_n^{(k)}(1-\tau)\}\|a_n - a^*\|,$$

for all $n \in N$ and $\{a_n\}_{n=0}^{\infty}$ strongly convergent to $a^* \in \mathfrak{F}(S) \cap \Omega(\mathfrak{C}, S, \mathfrak{T})$.

(ii) If $\{s_n\}_{n=0}^{\infty}$ sequence converges to $a^* \in \mathfrak{F}(S) \cap \Omega(\mathfrak{C}, S, \mathfrak{T})$, then the sequence $\{s_n - a_n\}_{n=0}^{\infty}$ strongly convergent to point 0 with the successive condition:

$$\|s_{n+1} - a_{n+1}\| \leq \tau^2\|s_n - a_n\| + (\tau^2 + 3\tau)\max_{\{k=1,2,3,4\}}\{1-\alpha_n^{(k)}(1-\tau)\}\|s_n - a^*\|$$

for every $n \in N$ and the sequence $\{s_n\}_{n=0}^{\infty}$ strongly convergent to the point $a^* \in \mathfrak{F}(S) \cap \Omega(\mathfrak{C}, S, \mathfrak{T})$.

Proof. (i) By above theorem (5), there we have $\lim_{n\to\infty}\|a_n-a^*\|=0$. We prove that $\lim_{n\to\infty}\|a_n-s_n\|=0$ and $\lim_{n\to\infty}\|s_n-a^*\|=0$. It depict from (4), (10), (11) and (12) that

$$
\begin{aligned}
&\|a_{n+1}-s_{n+1}\|\\
=&\|a_{n+1}-a^*+a^*-s_{n+1}\|\\
\le&\|a_{n+1}-a^*\|+\|a^*-s_{n+1}\|\\
=&(1-\alpha_n^{(3)})\|\Xi c_n-\Xi a^*\|+\alpha_n^{(3)}\|\Xi b_n-\Xi a^*\|+(1-\alpha_n^1)\|\Xi a^*-\Xi s_n\|+\alpha_n^1\|\Xi a^*-\Xi u_n\|\\
&+(1-\alpha_n^{(3)})\tau\|c_n-a^*\|+\alpha_n^{(3)}\tau\|b_n-a^*\|+(1-\alpha_n^{(1)})\tau\|a^*-s_n\|+\alpha_n^{(1)}\tau\|a^*-u_n\|\\
\le&(1-\alpha_n^{(3)})\tau^2\|a_n-a^*\|+\alpha_n^{(3)}\tau\{(1-\alpha_n^{(4)})\|a_n-a^*\|+\alpha_n^{(4)}\|c_n-a^*\|\}\\
&+(1-\alpha_n^{(1)})\tau\{\|a^*-a_n\|+\|a_n-s_n\|\}+\alpha_n^{(1)}\tau\{(1-\alpha_n^{(2)})\|a^*-s_n\|+\alpha_n^{(2)}\tau\|a^*-s_n\|\}\\
\le&(1-\alpha_n^{(1)})\tau\|a_n-s_n\|+((1-\alpha_n^{(3)})\tau^2\\
&+(1-\alpha_n^{(1)})\tau)\|a_n-a^*\|+\alpha_n^{(3)}\tau(1-\alpha_n^{(4)}(1-\tau))\|a_n-a^*\|\\
&+\alpha_n^{(1)}\tau((1-\alpha_n^{(2)}(1-\tau))\|a^*-s_n\|\\
\le&\ [(1-\alpha_n^{(1)})\tau+\alpha_n^{(1)}\tau(1-\alpha_n^{(2)}(1-\tau))]\|a_n-s_n\|
\end{aligned}
$$

(15)
$$
+[(1-\alpha_n^{(3)})\tau^2+(1-\alpha_n^{(1)})\tau+\alpha_n^{(3)}\tau(1-\alpha_n^{(4)}(1-\tau))+\alpha_n^{(1)}\tau(1-\alpha_n^{(2)}(1-\tau))]\|a_n-a^*\|
$$

Since $\{\alpha_n^{(k)}\}\subset[0,1]$ for all $k=1,2,3,4$ and $\tau<1$, then $1-\alpha_n^{(2)}(1-\tau)<1,(1-\alpha_n^{(3)})\le(1-\alpha_n^{(3)}(1-\tau))$ and $(1-\alpha_n^{(1)})\le(1-\alpha_n^{(1)}(1-\tau))$ for every $n\in N$

Making use of these inequalities in equation (15), we find

$$
\|a_{n+1}-s_{n+1}\|\le\tau\|a_n-s_n\|+(\tau^2+3\tau)\max_{\{k=1,2,3,4\}}\{1-\alpha_n^{(k)}(1-\tau)\}\|a_n-a^*\| \tag{16}
$$

Set

$$
\begin{aligned}
\Upsilon_n^{(1)}&=\|a_n-s_n\|\ge0\\
\nu&=\tau\in(0,1)\\
\Upsilon_n^{(2)}&=(\tau^2+3\tau)\max_{\{k=1,2,3,4\}}\{1-\alpha_n^{(k)}(1-\tau)\}\|a_n-a^*\| \text{ for every } n\in N.
\end{aligned}
$$

Since $\{\max_{\{k=1,2,3,4\}}\{1-\alpha_n^{(k)}(1-\tau)\}\}_{n=0}^{\infty}$ is "bounded", so is $\{(\tau^2+3\tau)\max_{k\in\{1,2,3,4\}}\{1-\alpha_n^{(k)}(1-\tau)\}\}_{n=0}^{\infty}$ and thus there is a constant $R>0$ such that

$$|(\tau^2+3\tau)\max_{k=1,2,3,4}\{1-\alpha_n^{(k)}(1-\tau)\}|<R\ \forall\ n\in N$$

Let $\varepsilon>0$ and $\zeta_n=\|a_n-a^*\|$ converges to point 0 and $\frac{\varepsilon}{R}>0$ thus there exists $n_0\in N$ such that $|\zeta_n|<\frac{\varepsilon}{R}\ \forall\ n\geq n_0$. Hence

$$|(\tau^2+3\tau)\max_{k=1,2,3,4}\{1-\alpha_n^{(k)}(1-\tau)\}|<\varepsilon\ \forall\ n\geq n_0.$$

As a result, we have $\lim_{n\to\infty}\Upsilon_n^{(2)}=0$ fulfils the condition of lemma (4) and we get, $\lim_{n\to\infty}\|a_n-s_n\|=0$. As $\lim_{n\to\infty}\|a_n-a^*\|=0$ along with

$$\|s_n-a^*\|\leq\|s_n-a_n\|+\|a_n-a^*\|,$$

we find that $\lim_{n\to\infty}\|s_n-a^*\|=0$.

(ii) Now, we show that $\lim_{n\to\infty}\|s_n-a_n\|=0$ and $\lim_{n\to\infty}\|s_n-a^*\|=0$. It depicts from (4), (10), (11) and (12) that

$$\begin{aligned}
&\|s_{n+1}-a_{n+1}\|\\
\leq&\|s_{n+1}-a^*\|+\|a^*-a_{n+1}\|\\
\leq&(1-\alpha_n^{(1)})\tau\|s_n-a^*\|+\alpha_n^{(1)}\tau\|u_n-a^*\|+(1-\alpha_n^{(3)})\tau\|a^*-c_n\|+\alpha_n^{(3)}\tau\|a^*-b_n\|\\
\leq&(1-\alpha_n^{(1)})\tau\|s_n-a^*\|+\alpha_n^{(1)}\tau\{(1-\alpha_n^{(2)})\|s_n-a^*\|+\alpha_n^{(2)}\tau\|s_n-a^*\|\}\\
&+(1-\alpha_n^{(3)})\tau^2\{\|a^*-s_n\|+\|s_n-a_n\|\}+\alpha_n^{(3)}\{(1-\alpha_n^{(4)})\|a^*-a_n\|+\alpha_n^{(4)}\|a^*-c_n\|\}\\
\leq&(1-\alpha_n^{(3)})\tau^2\|s_n-a_n\|+[(1-\alpha_n^{(1)})\tau+\alpha_n^{(1)}(1-\alpha_n^{(2)}(1-\tau))+(1-\alpha_n^{(3)}\tau^2)]\|s_n-a^*\|\\
&+\alpha_n^{(3)}\tau(1-\alpha_n^{(4)}(1-\tau))\|a^*-a_n\|\\
\leq&[(1-\alpha_n^{(3)})\tau^2+\alpha_n^{(3)}\tau(1-\alpha_n^{(4)}(1-\tau))]\|s_n-a_n\|
\end{aligned}$$

(17)

$$+[(1-\alpha_n^{(1)})\tau+\alpha_n^{(1)}(1-\alpha_n^{(2)}(1-\tau))+(1-\alpha_n^{(3)}\tau^2+\alpha_n^{(3)}\tau(1-\alpha_n^{(4)}(1-\tau))]\|s_n-a^*\|$$

Since $\{\alpha_n^{(k)}\}\subset[0,1]$ for all $k=1,2,3,4$ and $\tau<1$, then $1-\alpha_n^{(2)}(1-\tau)<1,(1-\alpha_n^{(3)})\leq(1-\alpha_n^{(3)}(1-\tau))$ and $(1-\alpha_n^{(1)})\leq(1-\alpha_n^{(1)}(1-\tau))$ for every $n\in N$, making use of these inequalities in equation (17), we find

$$\|s_{n+1}-a_{n+1}\| \leq \tau^2\|s_n-a_n\|+(\tau^2+3\tau)\max_{\{k=1,2,3,4\}}\{1-\alpha_n^{(k)}(1-\tau)\}\|s_n-a^*\| \tag{18}$$

Set

$$\Upsilon_n^{(1)} = \|a_n-s_n\| \geq 0$$

$$\nu = \tau^2 \in (0,1)$$

$$\Upsilon_n^{(2)} = (\tau^2+3\tau)\max_{\{k=1,2,3,4\}}\{1-\alpha_n^{(k)}(1-\tau)\}\|s_n-a^*\| \ \forall\ n \in N$$

Since $\max_{\{k=1,2,3,4\}}\{1-\alpha_n^{(k)}(1-\tau)\}$ is "bounded", so is $(\tau^2+3\tau)\max_{\{k=1,2,3,4\}}\{1-\alpha_n^{(k)}(1-\tau)\}$, thus there is a constant $M>0$ such that

$$|(\tau^2+3\tau)\max_{k=1,2,3,4}\{1-\alpha_n^{(k)}(1-\tau)\}| < M\ \forall\ n \in N$$

Let $\varepsilon>0$ and $\beta_n = \|s_n-a^*\|$ converges to point 0 and $\frac{\varepsilon}{M}>0\ \exists\ m_0 \in N$ in such a way that $|\beta_n| < \frac{\varepsilon}{M}\ \forall\ n \geq m_0$. Hence

$$|(\tau^2+3\tau)\max_{k=1,2,3,4}\{1-\alpha_n^{(k)}(1-\tau)\}| < \varepsilon\ \forall\ n \geq m_0.$$

As a result, we have $\lim_{n\to\infty}\Upsilon_n^{(2)}=0$ fulfils the condition of lemma (4) and we obtain, $\lim_{n\to\infty}\|s_n-a_n\|=0$. As $\lim_{n\to\infty}\|s_n-a^*\|=0$ and

$$\|a_n-a^*\| \leq \|a_n-s_n\|+\|s_n-a^*\|,$$

we find that $\lim_{n\to\infty}\|a_n-a^*\|=0$. □

Theorem 7. *Assume $\mathscr{H},\mathfrak{C},S,\mathfrak{T}$ and Ξ are defined in earlier theorem* (5) *with τ expressed in equation* (9). *Consider $\{s_n\}_{n=0}^{\infty}$ with $\{a_n\}_{n=0}^{\infty}$ as two iterations, defined by algorithm* (4) *and* (11)*, jointly, in which $\{\alpha_n^{(k)}\}_{n=0}^{\infty}$ as subset of [0,1] for every $k=1,2,3,4$ and $\{\alpha_n^{(1)}\}_{n=0}^{\infty}$ with $\{\alpha_n^{(3)}\}_{n=0}^{\infty}$ holds $\lim_{n\to\infty}\alpha_n^{(1)}=0$ and $\lim_{n\to\infty}\alpha_n^{(3)}=0$. If the circumstances defined in* (3) *remains the same then $\{a_n\}_{n=0}^{\infty}$ iteration converges faster and easier than $\{s_n\}_{n=0}^{\infty}$ iteration, to the point $a^* \in \mathfrak{F}(S)\cap\Omega(\mathfrak{C},S,\mathfrak{T})$ by assuming that $a_0=s_0$.*

Proof. From (13), we find

$$\|a_{n+1}-a^*\| = \tau^{n+1}\|a_0-a^*\|\prod_{i=0}^{n}[\tau+\alpha_i^{(3)}(1-\tau)-\alpha_i^{(3)}\alpha_i^{(4)}(1-\tau)] \text{ for all } n \in N. \tag{19}$$

Using (4) and (10), we find

$$\|s_{n+1}-a^*\| \geq (1-\alpha_n^{(1)})\tau\|s_n-a^*\| - \alpha_n^{(1)}\tau\|u_n-a^*\|$$
$$\geq (1-\alpha_n^{(1)})\tau\|s_n-a^*\| - \alpha_n^{(1)}\tau(1-\alpha_n^{(2)})\|s_n-a^*\| - \alpha_n^{(2)}\tau\|s_n-a^*\|$$
$$= [(1-\alpha_n^{(1)})\tau - \alpha_n^{(1)}\tau(1-\alpha_n^{(2)}(1-\tau))]\|s_n-a^*\| \quad (20)$$

As $(1-\alpha_n^{(1)}(1-\tau)) < 1$ for all $n \in N$, (20) becomes

$$\|s_{n+1}-a^*\| \geq (1-2\alpha_n^{(1)})\tau\|s_n-a^*\|$$
$$\geq \cdots$$
$$\geq \|a_0-a^*\|\tau\prod_{i=0}^{n}[(1-2\alpha_n^{(1)})] \text{ for all } n \in N. \quad (21)$$

From (19) and (21) with assumption $a_0 = s_0$, we obtain

$$\frac{\|a_{n+1}-a^*\|}{(\|s_{n+1}-a^*\|)} = \frac{(\tau^{n+1}\prod_{i=0}^{n}[\tau+\alpha_i^{(3)}(1-\tau)-\alpha_i^{(3)}\alpha_i^{(4)}(1-\tau)])}{\tau(\prod_{i=0}^{n}[1-2\alpha_i^{(1)}])}$$
$$= \frac{\tau^{n}\prod_{i=0}^{n}[\tau+\alpha_i^{(3)}(1-\tau)-\alpha_i^{(3)}\alpha_i^{(4)}(1-\tau)]}{\prod_{i=0}^{n}[1-2\alpha_i^{(1)}]} \text{ for all } n \in N.$$

Define

$$\phi_n = \frac{\tau^{n}\prod_{i=0}^{n}[\tau+\alpha_i^{(3)}(1-\tau)-\alpha_i^{(3)}\alpha_i^{(4)}(1-\tau)]}{\prod_{i=0}^{n}[1-2\alpha_i^{(1)}]} \ \forall\, n \in N$$

Since $\alpha_{n+1}^{(4)} \in [0,1]\ \forall\ n \in N$ and $\tau < 1$, then $\alpha_{n+1}^{(4)}(1-\tau)$ is "bounded" $\forall\ n \in N$. Hence by assumption $\lim_{n\to\infty}\alpha_n^{(1)} = 0 = \lim_{n\to\infty}\alpha_n^{(3)} = 0$, we calculated

$$\lim_{n\to\infty}\frac{\phi_{n+1}}{\phi_n} = \lim_{n\to\infty}\frac{\tau[\tau+\alpha_{n+1}^{(3)}(1-\tau)-\alpha_{n+1}^{(3)}\alpha_{n+1}^{(4)}(1-\tau)]}{[1-2\alpha_{n+1}^{(1)}]} = \tau^2 < 1.$$

By "ratio test", the series $\sum_{n=0}^{\infty}\phi_n$ absolutely converges. Thus

$$0 \leq \lim_{n\to\infty}\frac{\|a_{n+1}-a^*\|}{(\|s_{n+1}-a^*\|)} \leq \lim_{n\to\infty}\phi_n = 0.$$

As a result, from definition (1.1), $\{a_n\}_{n=0}^{\infty}$ converges faster and easier than $\{s_n\}_{n=0}^{\infty}$ to point $a^* \in \mathfrak{F}(S)\cap\Omega(\mathfrak{C},S,\mathfrak{T})$. □

Example 2.1. Consider $\mathscr{H} = \Re$ with $\mathfrak{C} = [0,1]$. As $\Re$ is a "Hilbert space" with the induced norm $\|x\| = |x|$ by "inner product" $\langle a,b\rangle = a.b$. Assume $T : [0,1] \to \Re$ and S be the same as defined in example 2.4 of [3]. $\mathfrak{T}$ is 4-Lipschitzian mapping with relaxed $(\frac{1}{17},1)$-cocoercive. We select $\sigma = 1/157$. As all the constants σ, μ, ν satisfies the condition (3).
If we consider $\alpha_n^{(1)} = \alpha_n^{(3)} = \frac{1}{n+3}, \alpha_n^{(2)} = \alpha_n^{(4)} = \frac{7n+3}{8n-3}$ for every $n \in N$ in (4) and (11) for S and $\mathfrak{T}$, then Table (1) along with Figure (1) shows the convergence of iterative algorithm (11) $\{a_n\}_{n=0}^{\infty}$ for $a_0 = s_0 = 0.01$ and compare its convergence with $\{s_n\}_{n=0}^{\infty}$, where we find that $\{a_n\}_{n=0}^{\infty}$ converges easily than $\{s_n\}_{n=0}^{\infty}$ to $a^* = 0 \in \mathfrak{F}(S) \cap \Omega(\mathfrak{C}, S, \mathfrak{T})$. $\alpha_n^{(1)} = \alpha_n^{(3)} = \frac{1}{n+3}$ and $\alpha_n^{(2)} = \alpha_n^{(4)} = \frac{7n+3}{8n-3}, \sigma = 1/157, S = \sin x$ and $\mathfrak{T}(a) = a^{(3)} + a$

TABLE 1. Compare the convergence rate of algorithm (4) with (11)

n	Algorithm (4)	Algorithm (11)
1	0.03	0.03
2	0.007571645	0.009290658
3	0.001908139	0.000625897
4	0.000480656	0.000101706
5	0.000121047	2.10889E-05
6	3.04794E-05	4.93255E-06
8	1.93188E-06	3.27996E-07
9	4.86318E-07	8.98379E-08
11	3.08132E-08	7.26813E-09
13	1.95E-09	6.30612E-10
15	1.23651E-10	5.74197E-11
16	3.11198E-11	1.75683E-11
17	7.83191E-12	5.41656E-12
18	1.97102E-12	1.68128E-12
⋮	⋮	⋮

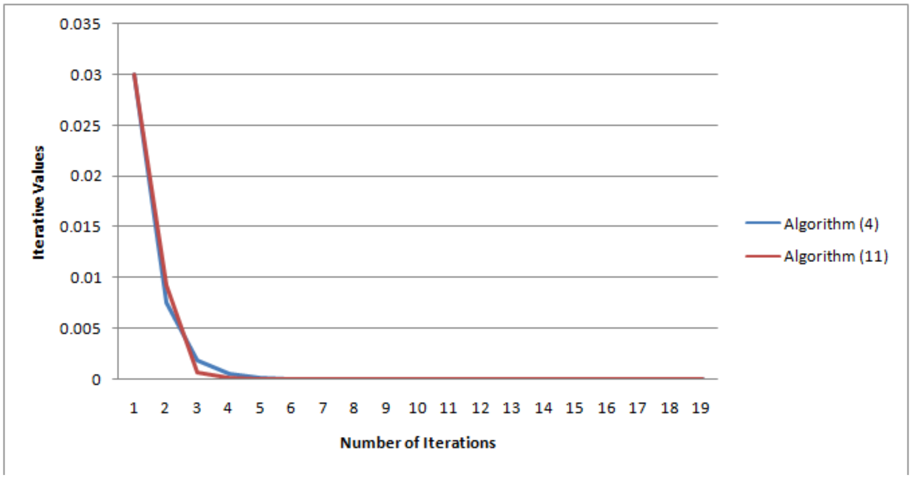

FIGURE 1. Plot the value of algorithm (4) with (11) for example (2.1)

3. APPLICATION: "CONVEX MINIMIZATION PROBLEM"

Consider a convex mapping $\mathfrak{f}:\mathfrak{C}\rightarrow\mathfrak{R}$ where $\mathfrak{C}$ is "closed and convex subset of Hilbert space $\mathscr{H}$". The problem of " convex minimization problem" is:

$$\min_{a\in\mathfrak{C}}\mathfrak{g}(x), \tag{22}$$

in which $\mathfrak{g}$ is Frechet differentiable and $\nabla\mathfrak{g}$ is gradient of $\mathfrak{g}$. As already known, a^* is solution of problem (22) only if the successive "variational inequality" hold:

$$a^*\in\mathfrak{C},\langle\nabla\mathfrak{g}a^*,a-a^*\rangle\geq 0 \text{ for all } a\in\mathfrak{C}.$$

it means, $a^*\in\Omega(\mathfrak{C},\mathfrak{T})$.
Equitably, a^* solves the situation (22) iff $a^*=P_{\mathfrak{C}}(a^*-\sigma\nabla\mathfrak{g}(a^*))$. Gradient projection technique is much fascinating tool for solving (22) and defined as:

$$a_{n+1}=P_{\mathfrak{C}}(a_n-\sigma\nabla\mathfrak{g}(a^*)) \tag{23}$$

Considering $S = I$ in the iterative algorithm (11) while $\mathfrak{T}$ is gradient of convex function $\mathfrak{g}$, then the accompanying iteration converges to the existing solution of problem (22):

$$\begin{aligned} & a_0 \in \mathfrak{C}, \\ & a_{n+1} = (1-\alpha_n^{(3)})P_{\mathfrak{C}}(c_n - \rho \nabla \mathfrak{g} c_n) + \alpha_n^{(3)} P_{\mathfrak{C}}(b_n - \rho \nabla \mathfrak{g} b_n) \\ & b_n = (1-\alpha_n^{(4)})a_n + \alpha_n^{(4)} c_n \\ & c_n = P_{\mathfrak{C}}(a_n - \rho \nabla \mathfrak{g} a_n) \ \forall\, n \in N. \end{aligned} \tag{24}$$

where $\alpha_n^{(3)}$ and $\alpha_n^{(4)} \subset [0,1]$.

Theorem 8. *Presume that the problem* (22)*can be solved. Consider a convex mapping* $\mathfrak{g} : \mathfrak{C} \to \mathfrak{R}$*, having its "gradient relaxed* (ν, r)*-cocoercive and* ν*-Lipschitzian mapping" from* $\mathfrak{C}$ *to* $\mathscr{H}$ *and satisfied condition* (3)*. Let* $\{a_n\}$ *be the iteration defined in equation* (24) *with* $\alpha_n^{(3)}$ *and* $\alpha_n^{(4)} \subset [0,1]$*, then* $\{a_n\}$ *converges to* a^* *which is also solution of* (22) *with the accompanying assumption:*

$$\|a_{n+1} - a^*\| \leq \tau^{n+1} \|a_0 - a^*\| \text{ for all } n \in N,$$

in which τ *expressed by* (9).

Proof. By assuming $\mathfrak{T} = \nabla \mathfrak{g}$ and $S = I$ in the above theorem (5). As I is trivially nonexpansive. Proceeding according to theorem (5), we obtain that
$a^* \in \mathfrak{F}(S) \cap \Omega(\mathfrak{C}, \S, \mathfrak{T}) = \Omega(\mathfrak{C}, \mathfrak{T}) = \{a \in \mathfrak{C} : \langle \mathfrak{T}a, b-a \rangle \geq 0, \forall b \in \mathfrak{C}\}$ which gives us that point a^* is the result of problem (22). □

Example 3.1. Consider $\mathscr{H}, \mathfrak{g}, \nabla \mathfrak{g}(f) = 3f$ from the example 2.6 of [3]. For this situation it is not difficult to see that the problem (24) converges to 0 for any $\{\alpha_n^{(3)}\}_{n=0}^{\infty}, \{\alpha_n^{(4)}\}_{n=0}^{\infty} \subset [0,1]$. Take $\alpha_n^{(3)} = \alpha_n^{(4)} = 1/(n+3)$ and $\sigma = 1/6$, then we have

$$\begin{aligned} a_{n+1} &= \left(1 - \frac{1}{n+3}\right) P_{\mathfrak{C}}\left(\frac{c_n}{2}\right) + \alpha_n^{(3)} P_{\mathfrak{C}}\left(\frac{b_n}{2}\right) \\ b_n &= \left(1 - \frac{1}{n+3}\right) a_n + \frac{1}{n+3} c_n \\ c_n &= P_{\mathfrak{C}}\left(\frac{a_n}{2}\right) \end{aligned} \tag{25}$$

where

$$P_{\mathfrak{C}}(x) = \frac{x}{2} \text{ for all } x \in \mathfrak{C}.$$

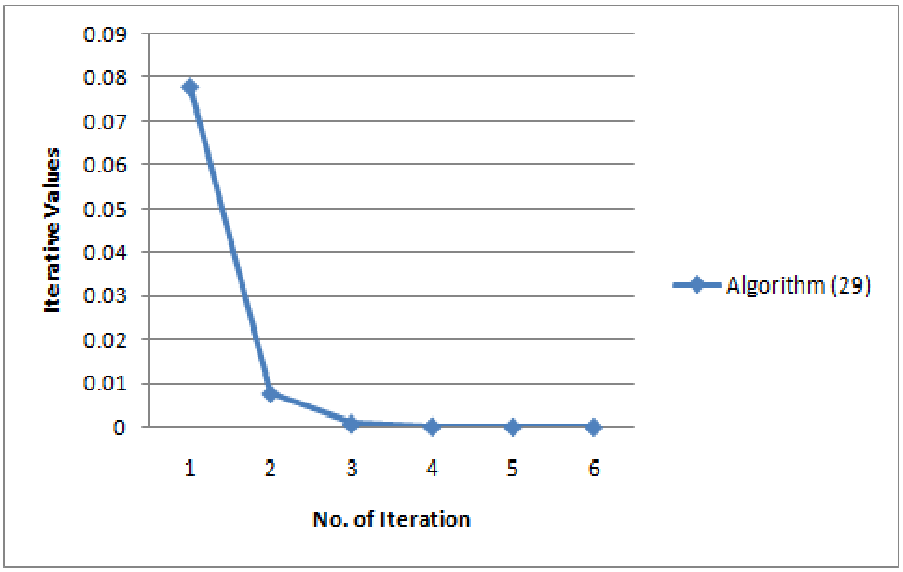

FIGURE 2. Convergence behaviour of $\|a_{n+1} - 0\|_2$ for initial guess $a_0(a) = a$

TABLE 2. Convergence behaviour of algorithm (25)

n	Algorithm (25)	$\|a_{n+1} - 0\|_2$
0	a	0.077790953
1	$\frac{5a}{48}$	0.007596773
2	$\frac{125a}{12288}$	0.000702701
3	$\frac{185a}{196608}$	6.22184E-05
4	$\frac{3145a}{37748736}$	5.31713E-06
5	$\frac{210715a}{29565009024}$	1.92123E-07
$\vdots$	$\vdots$	$\vdots$

Table (2) along with the Figure (2), shows the convergence of algorithm (25) to point 0 for the starting point $a_0(a) = a$..

4. Conclusion

In Theorem (6), we obtain the convergence results of two iteartive schemes generated by (4) and (11). Theorem (7) compare the convergence of algorithm and showed that algorithm (11) converges faster than algorithm (4). This technique is same as used by Erturk [3] in 2021.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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