Malaya Journal of Matematik, Vol. 7, No. 1, 104-107, 2019

https://doi.org/10.26637/MJM0701/0020

On restrained domination number of some wheel related graphs

S. K. Vaidya[1]* and P. D. Ajani[2]

Abstract

For a graph $G = (V,E)$, a set $S \subseteq V$ is a restrained dominating set if every vertex not in S is adjacent to a vertex in S and also to a vertex in $V - S$. The minimum cardinality of a restrained dominating set of G is called restrained domination number of G, denoted by $\gamma_r(G)$. We investigate restrained domination number of some wheel related graphs.

Keywords

Dominating set, restrained dominating set, restrained domination number.

AMS Subject Classification

05C69.

[1] *Department of Mathematics, Saurashtra University, Rajkot - 360 005, Gujarat, India*

[2] *Department of Mathematics, Atmiya University, Rajkot-360005, Gujarat, India.*

***Corresponding author**: [1] samirkvaidya@yahoo.co.in; [2] paragajani@gmail.com

Article History: Received **12** September **2018**; Accepted **29** January **2019**

Contents

1. Introduction

The concept of domination in graphs is one of the most rapidly developing areas within and outside of graph theory. It has motivated many researchers to work on it due to its diversified applications and its potential to handle real life situations.

All the graphs considered here are finite, connected and undirected without loops and multiple edges. For a graph $G = (V,E)$, the number of vertices and edges of G is called the order and size of G respectively. The cardinality of vertex set V and edge set E are denoted by $|V|$ and $|E|$ respectively. The minimum degree among the vertices of G is denoted by $\delta(G)$ while the maximum degree among the vertices of G is denoted by $\triangle(G)$.

A set $S \subseteq V$ is a dominating set if every vertex $v \in V - S$ is either an element of S or is adjacent to an element of S. A γ-set is a dominating set of minimum cardinality. The domination number $\gamma(G)$ is a minimum cardinality of γ-sets. A brief account of dominating sets and its related concepts can be found in Haynes *et al* [7]. Many variants of domination are introduced as a combination of two different dominating parameters. Total domination [2], equitable domination [10], global domination [9], independent domination [1, 8] are among worth to mention.

The present work is focused on one such variant known as restrained domination in graphs. A set $S \subseteq V$ is a restrained dominating set if every vertex not in S is adjacent to a vertex in S as well as to a vertex in $V - S$. The minimum cardinality of a restrained dominating set S is called the restrained domination number of G which is denoted by $\gamma_r(G)$. It is obvious that all mutually non-adjacent vertices must belong to every restrained dominating set. The concept of restrained domination was introduced by Telle and Proskurowski [11] as a vertex partitioning problem. An application of restrained domination is that of prisoners and guards. For security, each prisoner must be seen by some guard; the concept is that of domination. However, in order to protect the rights of prisoners, we may also require that each prisoner is seen by another prisoner; the concept is that of restrained domination.

The concept of restrained domination in the context of path and cycle is studied by Vaidya and Ajani in [12, 13]. The restrained domination in trees is well studied in [3, 6] while the restrained domination in complete graphs, multipartite graphs and the graphs with minimum degree two is well explored by Domke *et al* [4, 5]. In the present work we investigate restrained domination number of some wheel related graphs.

Definition 1.1. *The wheel graph W_n is defined to be the join $K_1 + C_n$. The vertex corresponding to K_1 is known as the apex and the vertices corresponding to cycle C_n are known as rim vertices while the edges corresponding to cycle are known as rim edges.*

Definition 1.2. *The helm H_n is the graph obtained from wheel W_n by attaching a pendant edge to each rim vertex.*

Definition 1.3. *The closed helm CH_n is the graph obtained from helm H_n by joining each pendant vertex to form a cycle.*

Definition 1.4. *The web graph $W(t,n)$ is the graph obtained by joining the pendant vertices of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle. $W(t,n)$ is the generalized web with t cycles each of order n.*

Definition 1.5. *The flower Fl_n is the graph obtained from helm H_n by joining each pendant vertex to the apex of helm H_n.*

Definition 1.6. *The sunflower Sf_n is the graph obtained from flower Fl_n by attaching n pendant edges to the apex of flower Fl_n.*

We state following obvious results without proofs.

Proposition 1.7. For cycle C_n, $\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

Proposition 1.8. For the wheel W_n for $n \geq 3$, $\gamma_r(W_n) = 1$.

2. Main Results

Theorem 2.1. For the helm H_n, for $n \geq 3$

$$\gamma_r(H_n) = n + 1.$$

Proof: Let $v_1, v_2, ..., v_n$ be the pendant vertices, $u_1, u_2, ..., u_n$ be the rim vertices and v be the apex of maximum degree n of helm H_n with $|V(H_n)| = 2n + 1$.

The pendant vertices $v_1, v_2, ..., v_n$ are mutually non-adjacent which must be in every restrained dominating set. Moreover pendant vertices $v_1, v_2, ..., v_n$ dominate the rim vertices $u_1, u_2, ..., u_n$. A restrained dominating set S should contain pendant vertices $v_1, v_2, ..., v_n$ and the apex v will be dominated by itself. If possible, suppose $v \notin S$ then it is not dominated by any vertex. Therefore, $v \in S$ which implies that $|S| = n + 1$.

Note that $S = \{v_1, v_2, ..., v_n, v\}$ is a restrained dominating set with minimum cardinality because removal of any of the vertices from set S will not dominate all the vertices of CH_n. Moreover every vertex in $V - S$ is adjacent to vertex in S and to a vertex in $V - S$. Hence $\gamma_r(H_n) = n + 1$.

Illustration 2.2. The helm H_4 is shown in Figure 1 where the set of solid vertices is its restrained dominating set of minimum cardinality.

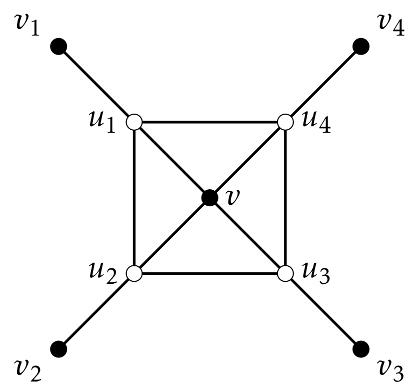

Figure 1: $\gamma_r(H_4) = 5$

Theorem 2.3. For the closed helm CH_n, for $n \geq 3$

$$\gamma_r(CH_n) = \begin{cases} 2, & \text{for } n = 4 \\ \left\lceil \frac{n}{3} \right\rceil + 1, & \text{for } n \geq 3 - \{4\}. \end{cases}$$

Proof: The closed helm CH_n contains wheel W_n and the outer cycle C_n. Let v denote the apex of wheel. Also $u_1, u_2, ..., u_n$ be the rim vertices of wheel W_n of CH_n and $v_1, v_2, ..., v_n$ be the corresponding adjacent vertices of outer cycle of CH_n. So $|V(CH_n)| = 2n + 1$.

For $n = 4$, $|V(CH_4)| = 9$ and $\Delta(CH_4) = 4$. It follows that at least two vertices are required to dominate all the vertices of CH_4. If $S \subseteq V(CH_4)$ is a restrained dominating set then $|S| = 2$, which is minimum. Therefore $\gamma_r(CH_4) = 2$.

For $n \geq 3 - \{4\}$, $deg(v) = n = \Delta(CH_n)$ and by Proposition 1.8, apex v dominates all the vertices of W_n. If $S \subseteq V(CH_n)$ is a restrained dominating set then it must contain the apex v. Due to adjacency nature of vertices $v_1, v_2, ..., v_n$ of outer cycle with corresponding vertices $u_1, u_2, ..., u_n$ of W_n, by Proposition 1.7 at least $\left\lceil \frac{n}{3} \right\rceil$ vertices are required to dominate all the remaining vertices of outer cycle of CH_n. It follows that, $|S| = \left\lceil \frac{n}{3} \right\rceil + 1$.

If possible, suppose S' is a restrained dominating set such that $|S'| = \left\lceil \frac{n}{3} \right\rceil < |S|$. Now $\Delta(CH_n) = n$ and in order to attain minimum cardinality, S' can not contain the vertices where each vertex among them can dominate distinct n vertices of CH_n. Moreover $\left\lceil \frac{n}{3} \right\rceil .\Delta(CH_n) = \left\lceil \frac{n}{3} \right\rceil .n < 2n + 1 = |V(CH_n)|$. Therefore S' can not be a restrained dominating set of CH_n. This implies that S is a restrained dominating set with minimum cardinality for CH_n. Hence, $\gamma_r(CH_n) = \left\lceil \frac{n}{3} \right\rceil + 1$.

Illustration 2.4. The closed helm CH_5 is shown in Figure 2 where the set of solid vertices is its restrained dominating set of minimum cardinality..

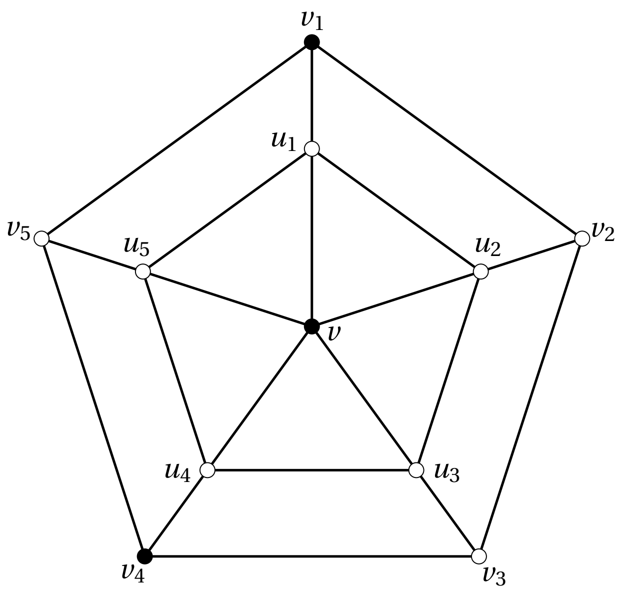

Figure 2: $\gamma_r(CH_5) = 3$

Theorem 2.5. For the generalized web graph $W(t,n)$,

$$\gamma_r(W(t,n)) = (t-2)\left\lceil \frac{n}{3} \right\rceil + (n+1).$$

Proof: Generalized web graph contains t cycles, each of order n. Let $u_1^1, u_2^1, u_3^1, ..., u_n^1$ be the vertices of innermost cycle of $W(t,n)$. Then denote the vertices $u_1^2, u_2^2, u_3^2, ..., u_n^2$ adjacent to $u_1^1, u_2^1, u_3^1, ..., u_n^1$ on the second cycle respectively. In general, denote the vertices for t cycles of $W(t,n)$ as $u_1^i, u_2^i, u_3^i, ..., u_n^i$, where $1 \leq i \leq t$. Let $v_1, v_2, v_3, ..., v_n$ be the pendant vertices and v be the apex of $W(t,n)$. Then $|V(W(t,n))| = n(t+1)+1$.

Note that $deg(v) = n = \Delta(W(t,n))$ and pendent vertices $v_1, v_2, v_3, ..., v_n$ are mutually non adjacent. Moreover apex v dominates it self as well as all the vertices of innermost cycle of $W(t,n)$ and pendant vertices $v_1, v_2, v_3, ..., v_n$ dominate themselves as well as all the vertices of outermost cycle of $W(t,n)$. Every restrained dominating set of $W(t,n)$ must contain the apex v and pendant vertices $v_1, v_2, v_3, ..., v_n$. Now, by Proposition 1.7 at least $\left\lceil \frac{n}{3} \right\rceil$ vertices are required for each cycle. Which implies that, at least $(t-2)\left\lceil \frac{n}{3} \right\rceil$ vertices are required to dominate all the vertices of remaining $(t-2)$ cycles of $W(t,n)$.

If $S \subseteq V(W(t,n))$ is a restrained dominating set then $|S| = (t-2)\left\lceil \frac{n}{3} \right\rceil + n + 1$, the set S is of minimum cardinality because removal of any of the vertices from set S will not dominate all the vertices of $W(t,n)$. Moreover every vertex of $V-S$ is adjacent to a vertex in S and also to a vertex in $V-S$. Hence $\gamma_r(W(t,n)) = (t-2)\left\lceil \frac{n}{3} \right\rceil + (n+1)$.

Illustration 2.6. The web graph $W(3,6)$ is shown in Figure 3 where the set of solid vertices is its restrained dominating set of minimum cardinality.

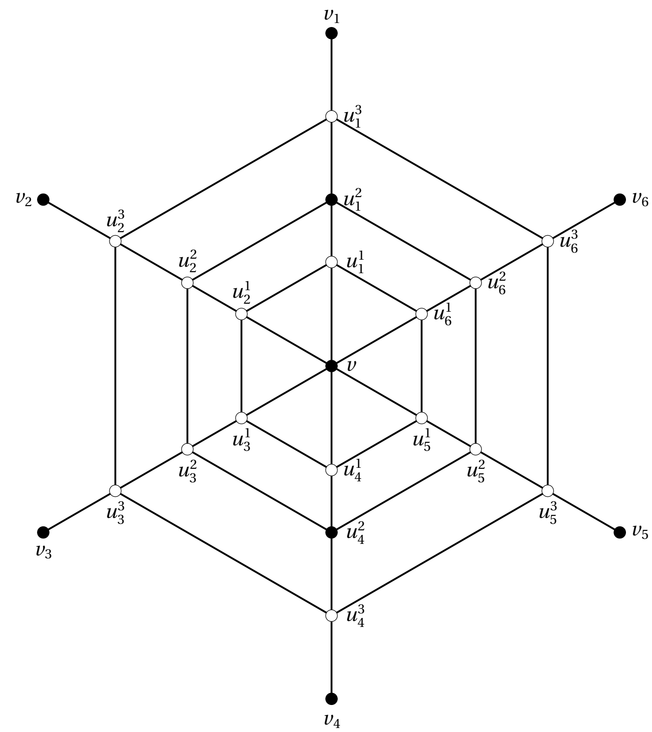

Figure 3: $\gamma_r(W(3,6)) = 9$

Theorem 2.7. For the sunflower Sf_n $(n \geq 3)$,

$$\gamma_r(Sf_n) = n+1.$$

Proof: Let v be apex and $v_2, v_4, v_6, ..., v_{2i}$, for $i = 1,2,3,...,n$ be the pendant vertices of Sf_n with $|V(Sf_n)| = 3n+1$. The apex v dominates $2n$ distinct vertices and pendant vertices are mutually non-adjacent.

It is very clear to see that, any restrained dominating set S must contain the apex and pendant vertices. Implies that $|S| = n+1$, which is of minimum cardinality. Hence, $\gamma_r(Sf_n) = n+1$.

Illustration 2.8. The sunflower Sf_4 is shown in Figure 4 where the set of solid vertices is its restrained dominating set of minimum cardinality.

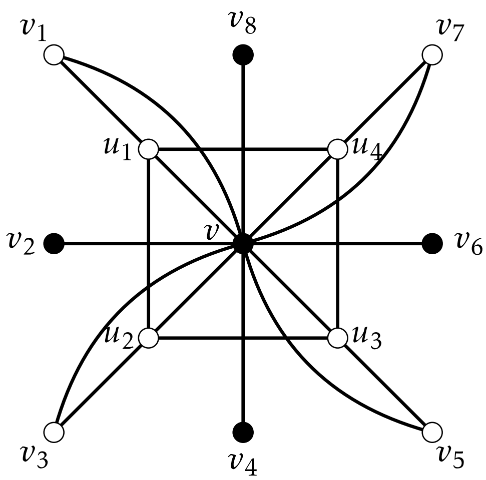

Figure 4: $\gamma_r(Sf_4) = 5$

Conclusion

The restrained domination of some standard graphs are already known, while we have investigated the restrained domination number of the graphs obtained from wheel W_n by means of various graph operations.

Acknowledgment

The authors are highly indebted to the anonymous referee for their kind comments and constructive suggestions on the first draft of this paper.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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