Open Packing Number of Some Path Related Graphs

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Abstract— **A subset *S* of vertices of *G* is an open packing of *G* if the open neighborhoods of the vertices of *S* are pairwise disjoint in *G* while open packing number of *G* is the maximum cardinality among all the open packing sets of *G*. We investigate open packing number of some graphs obtained from path.**

Keywords— **Neighborhood, Packing, Open Packing**
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I. INTRODUCTION

We begin with the finite, undirected and simple graph $G = (V(G), E(G))$. The open neighborhood of $v \in V(G)$ is $N(v) = \{u \in V(G)/uv \in E(G)\}$ and the closed neighborhood of $v \in V(G)$ is $N[v] = N(v) \cup \{v\}$. For any real number n, $\lceil n \rceil$ denotes the smallest integer not less than that n and $\lfloor n \rfloor$ denotes the greatest integer not greater than that n.

A packing of a graph G is a set of vertices whose closed neighborhoods are pairwise disjoint. The maximum cardinality of a packing set of G is called the packing number and it is denoted by $\rho(G)$. This concept was introduced by Biggs [1].

A subset S of $V(G)$ is an open packing of G if the open neighborhoods of the vertices of S are pairwise disjoint in G. The maximum cardinality of an open packing set is called the open packing number and is denoted by ρ^o. This concept was introduced by Henning and Slater [5]. A brief account of on open packing and its related concepts can be found in [2,4,6].

Proposition 1.1 [3] The inequality $\rho(G) \leq \rho^o(G) \leq 2\rho(G)$ hold for any graph G.

Definition 1.2 The square of a graph G denoted by G^2 has the same vertex set as of G and two vertices are adjacent in G^2 if they are at distance of 1 or 2 apart in G.

Definition 1.3 The switching of a vertex v of G means removing all the edges incident to v and adding edges joining v to every vertex which is not adjacent to v in G. We denote the resultant graph by $\tilde{G}$.

Definition 1.4 Let $G = (V(G), E(G))$ be a graph with $V(G) = S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_t \cup T$, where each S_i is a set of all the vertices having same degree (at least 2 vertices) and $T = V(G) \setminus \cup_{i=1}^{t} S_i$. The degree splitting graph $DS(G)$ is obtained from G by adding vertices $w_1, w_2, w_3, \ldots, w_t$ and joining to each vertex of S_i for $1 \leq i \leq t$.

Definition 1.5 The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G, say G' and G''. Join each vertex u' in G' to the neighbors of the corresponding vertex u'' in G''.

Definition 1.6 The Cartesian product of G and H is a graph, denoted as $G \times H$, whose vertex set $V(G) \times V(H)$. Two vertices (g,h) and (g',h') are adjacent precisely if $g=g'$ and $hh' \in E(H)$, or $gg' \in E(H)$ and $h=h'$. Thus, $V(G \times H) = \{(g,h)/\ g \in V(G)$ and $h \in V(H)\}$ and $E(G \times H) = \{(g,h)(g',h')/\ g = g'$, $hh' \in E(H)$ or $gg' \in E(G), h = h'\}$.

Definition 1.7 The ladder graph L_n is defined as $P_2 \times P_n$.

For any undefined term and notations in graph theory we refer to West [7] and Haynes *et al.* [3].

II. Main Results

Theorem 2.1 $\rho^o(P_n^2) = \left\lceil \frac{n}{5} \right\rceil; n \geq 3$

Proof: Let $V(P_n^2) = V(P_n) = \{v_1, v_2, \dots, v_n\}$ be the vertex set where $d_{P_n^2}(v_1) = d_{P_n^2}(v_n) = 2$, $d_{P_n^2}(v_2) = d_{P_n^2}(v_{n-1}) = 3$ and $d_{P_n^2}(v_i) = 4$, for all $i \in \{3,4,\dots,n-2\}$.

If S is any open packing set of P_n^2 then it is obvious that v_1 must belong to S as $d_{P_n^2}(v_1) = 2 = \delta(P_n^2)$.

We construct a set S of vertices as follows:

$$S = \left\{ v_{5i+1} / \ 0 \leq i \leq \left\lfloor \frac{n}{5} \right\rfloor \right\}$$

Then $|S| = \left\lceil \frac{n}{5} \right\rceil$. Moreover S is an open packing set of P_n^2 as $N(v) \cap N(u) = \emptyset$ for all $u, v \in S$. Further S is a maximal open packing set of P_n^2 because for any $w \in V(P_n^2) - S$, $N(v) \cap N(w) \neq \emptyset$ and $N(u) \cap N(w) \neq \emptyset$. Therefore, any superset containing the vertices greater than that of $|S|$can not be an open packing set of P_n^2.

Hence, $\rho^o(P_n^2) = \left\lceil \frac{n}{5} \right\rceil; n \geq 3$

Illustration 2.2 The graph P_n^2 and its open packing number is shown in Figure 1.

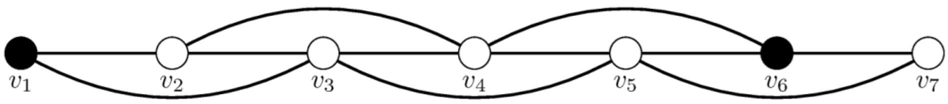

Figure 1: $\rho^o(P_n^2) = 2$

Theorem 2.3 $\boldsymbol{\rho^o(\widetilde{P_n})} = \begin{cases} 2 \text{ ; if } v \text{ is terminal vertex in } P_n \\ 3 \text{ ; if } v \text{ is internal vertex in } P_n \end{cases}$

Proof: Let $\widetilde{P_n}$ be a graph obtained by switching of an arbitrary vertex v_i of P_n. Let $V(\widetilde{P_n}) = V(P_n) = \{v_1, v_2, \dots, v_n\}$ be the vertex set. Suppose S is any open packing set of $\widetilde{P_n}$.

To prove this result we consider the following cases:

Case I: If the either of the terminal vertex is switched.

Without loss of generality we switch the vertex v_1 then it is obvious that $v_1 \notin S$ as $v_1 \in \bigcup_{i=2}^{n} N(v_i)$.

We construct a set S of vertices as follows:

$S = \{v_2, v_3\}$. Then $|S| = 2$. Moreover S is an open packing set of $\widetilde{P_n}$ as $N(v_2) \cap N(v_3) = \phi$. Further S is a maximal open packing set of $\widetilde{P_n}$ because for any $w \in V(\widetilde{P_n}) - S$, $N(w) \cap N(v_2) \neq \phi$ and $N(w) \cap N(v_3) \neq \phi$. Therefore any super set containing the vertices greater than that of $|S|$ can not be an open packing set of $\widetilde{P_n}$. Hence $\rho^o(\widetilde{P_n}) = 2$; if either of the terminal vertex is switched.

Case II: If the either of the internal vertex is switched.

Suppose v_i is switched vertex for all $i \in \{2,3,\dots,n-1\}$. We construct a set S of vertices as follows:

$S = \{v_{i-1}, v_{i+1}, v_{i+2}\}$ for any $i \in \{2,3,\dots,n-1\}$. Then $|S| = 3$. Moreover S is an open packing set of $\widetilde{P_n}$ as $N(v_{i-1}) \cap N(v_{i+2}) = \phi$ for any $i \in \{2,3,\dots,n-1\}$. Further S is a maximal open packing set of $\widetilde{P_n}$ because for any $w \in V(\widetilde{P_n}) - S$, $N(w) \cap N(v_{i+1}) \neq \phi$. Therefore any super set containing the vertices greater than that of $|S|$ can not be an open packing set of $\widetilde{P_n}$. Hence $\rho^o(\widetilde{P_n}) = 3$; if either of the terminal vertex is switched.

Illustration 2.4 The graph $\widetilde{P_7}$ and its open packing number is shown in Figure 2.

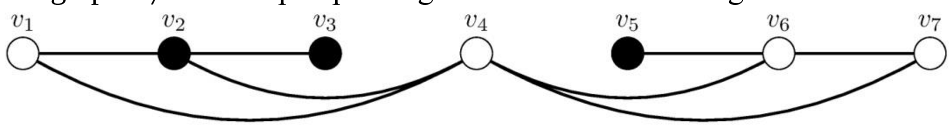

Figure 2: $\rho^o(\widetilde{P_7}) = 3$

Theorem 2.5 $\rho^o(DS(P_n)) = \begin{cases} 2; & \text{if } 3 \le n \le 5 \\ 3; & \text{if } n \ge 6 \end{cases}$

***Proof*:** The path P_n have two vertices of degree one and remaining $n-2$ vertices of degree two. Then $V(P_n) = \{v_i/1 \le i \le n\} = S_1 \cup S_2$ where $S_1 = \{v_1, v_n\}$ and $S_2 = \{v_i/2 \le i \le n-1\}$. To obtain $DS(P_n)$ from P_n, add two vertices x and y corresponding to S_1 and S_2 respectively. Thus $V(DS(P_n)) = V(P_n) \cup \{x, y\}$ and $E(DS(P_n)) = E(P_n) \cup \{xv_i/v_i \in S_1\} \cup \{yv_j/v_i \in S_2\}$.

Case I: For $n = 4, 5$

We construct a set S of vertices as follows:

$S = \{v_1, v_2\}$. Then $|S| = 2$. Moreover S is an open packing set of $DS(P_n)$ as $N(v_1) \cap N(v_2) = \phi$, Further S is a maximal open packing set of $DS(P_n)$ because for any $w \in V(DS(P_n)) - S$, $N(v_1) \cap N(w) \ne \phi$ and $N(v_2) \cap N(w) \ne \phi$. Therefore containing the vertices greater than that of $|S|$ can not be an open packing set of $DS(P_n)$. Hence $\rho^o(DS(P_n)) = 2$, for $n = 4, 5$.

Case II: For $n \ge 6$

We construct a set S of vertices as follows:

$S = \{x, v_1, v_4\}$. Then $|S| = 3$. Moreover S is an open packing set of $DS(P_n)$ as $N(v_1) \cap N(v_{42}) = \phi$, $N(x) \cap N(v_1) = \phi$ and $N(x) \cap N(v_4) = \phi$. Further S is a maximal open packing set of $DS(P_n)$ because for any $w \in V(DS(P_n)) - S$, $N(u) \cap N(w) \ne \phi$, $\forall u \in S$. Therefore containing the vertices greater than that of $|S|$ can not be an open packing set of $DS(P_n)$. Hence $\rho^o(DS(P_n)) = 3$, for $n \ge 6$.

Illustration 2.6 The graph $DS(P_7)$ and its open packing number is shown in Figure 3.

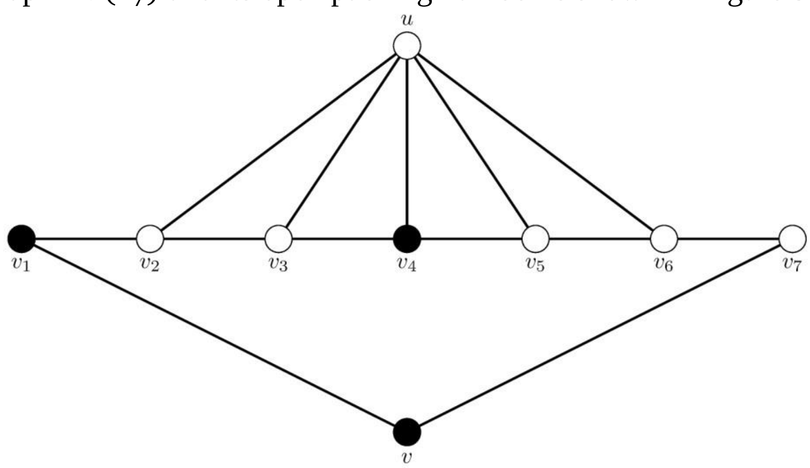

Figure 3: $\rho^o(DS(P_7)) = 3$

Theorem 2.7 $\rho^o(D_2(P_n)) = \begin{cases} \frac{n}{2}; & \text{if } n \equiv 0 (\text{mod } 4) \\ \left\lfloor \frac{n}{2} \right\rfloor + 1; & \text{otherwise} \end{cases}$

Proof: Let $D_2(P_2)$ be the shadow graph of P_n. Let $V(D_2(P_n)) = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n\}$ be the vertex set.

If S is any open packing set of $D_2(P_n)$ then it is obvious that v_1 must belong to S as $d_{D_2(P_n)}(v_1) = 2 = \delta(D_2(P_n))$.

We construct a set S of vertices as follows:

$$S = \left\{v_{8i+1}, v_{8i+2}, v_{8i+6}, u_{8i+5} / 0 \le i \le \left\lfloor \frac{n}{8} \right\rfloor\right\}$$

Then

$$|S| = \begin{cases} \frac{n}{2}; & \text{if } n \equiv 0 (\text{mod } 4) \\ \left\lfloor \frac{n}{2} \right\rfloor + 1; & \text{otherwise} \end{cases}$$

Moreover S is an open packing set of $D_2(P_n)$ as $N(v) \cap N(u) = \phi$ for all $u, v \in S$. Further S is a maximal open packing set of $D_2(P_n)$ because for any $w \in V(D_2(P_n)) - S$, $N(v) \cap N(w) \neq \phi$. Therefore, any superset containing the vertices greater than that $|S|$ can not be an open packing set $D_2(P_n)$.

Hence, $\rho^o(D_2(P_n)) = \begin{cases} \frac{n}{2}; & \text{if } n \equiv 0 (\text{mod } 4) \\ \left\lfloor \frac{n}{2} \right\rfloor + 1; & \text{otherwise} \end{cases}$

Illustration 2.8 The graph $D_2(P_7)$ and its open packing number is shown in Figure 4.

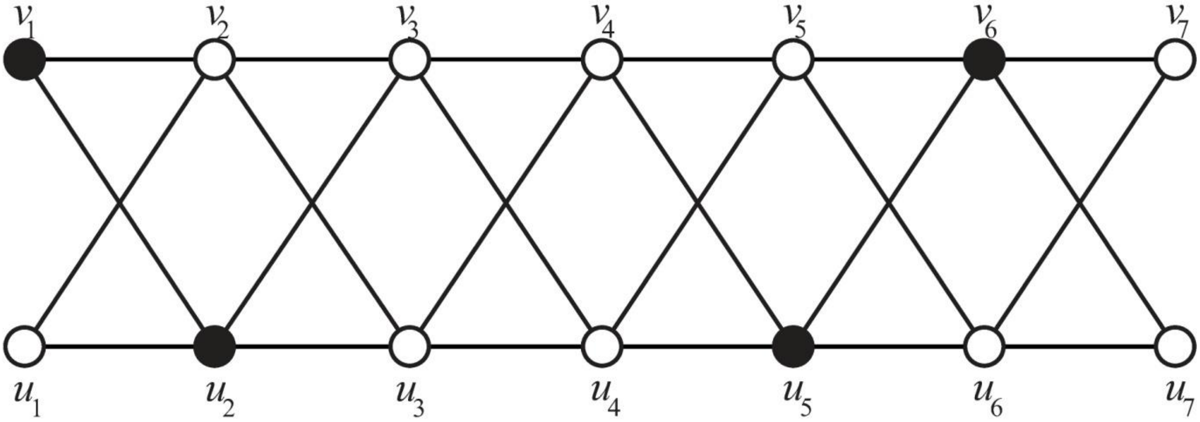

Figure 4: $\rho^o(D_2(P_7)) = 4$

Theorem 2.9 $\rho^o(L_n) = 2\left\lfloor \frac{n+2}{3} \right\rfloor$

Proof: Let $V(L_n) = \{u_i, v_i / 1 \le i \le n\}$ be the vertex set with $|V(L_n)| = 2n$, where $d_{L_n}(u_i) = d_{L_n}(v_i) = 3$, for all $i \in \{2,3,\dots,n-1\}$.

If S is any open packing set of L_n then it is obvious that v_1 and u_1 must belong to S as $d_{L_n}(v_1) = d_{L_n}(u_1) = 2 = \delta(L_n)$.

We construct a set S of vertices as follows:

$$S = \left\{v_{3i+1}, u_{3i+1} / 0 \le i \le \left\lceil \frac{n}{3} \right\rceil\right\}$$

Then $|S| = 2\left\lfloor\frac{n+2}{3}\right\rfloor$. Moreover S is an open packing set of L_n as $N(v) \cap N(u) = \phi$, for all $u, v \in S$. Further S is a maximal open packing set of L_n because for any $w \in V(L_n) - S$, $N(v) \cap N(w) \neq \phi$ and $N(u) \cap N(w) \neq \phi$. Therefore, any superset containing the vertices greater than that of $|S|$ can not be an open packing set of L_n.

Hence, $\rho^o(L_n) = 2\left\lfloor\frac{n+2}{3}\right\rfloor$.

Illustration 2.10 The graph L_7 and its open packing number is shown in Figure 5.

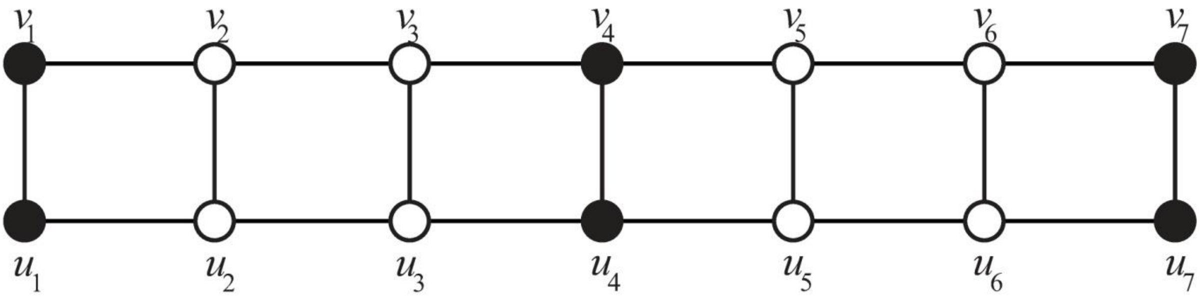

Figure 5: $\rho^o(L_7) = 6$

III. Conclusions

The open packing number of some standard graph families are known while we investigate open packing number of the larger graphs obtained from path P_n by means of some graph operations like degree splitting, and square of a graph, switching of a vertex, degree splitting, shadow graph of P_n and ladder.

Acknowledgment

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