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SOME NEW RESULTS ON CHROMATIC TRANSVERSAL DOMINATION IN GRAPHS

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Abstract

A vertex dominating set D of $V(G)$ is called a chromatic transversal dominating set of G if D intersects every color class of G. The minimum cardinality of D is called a chromatic transversal domination number of G. In this work we contribute some new results on chromatic transversal domination.

1. Introduction

We consider simple, finite, undirected and connected graph $G = (V(G), E(G))$. We denote the degree of a vertex v in a graph G by $d_G(v)$. The maximum degree among the vertices of G is denoted by $\Delta(G)$. For any real number n, $\lceil n \rceil$ denotes the smallest integer not less than n and $\lfloor n \rfloor$ denotes the greatest integer not greater than n. For the various graph theoretic notation and terminology we follow West [7]. For standard

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terminology and terms related to coloring are used in the sense of Zhang [9] while for any undefined terms related to the concept of domination we refer to Haynes et al. [1].

The study of graph coloring and its related concepts are getting momentum due to its diversified applications for the solution of many real life problems such as scheduling time-table, compiler register allocation, assigning mobile and radio frequency, etc. An excellent discussion on theory of graph coloring is carried out by Zhang [9].

An independent set of vertices in a graph G is a set of pairwise non-adjacent vertices of G. A proper k - coloring of a graph G is a function $f : V(G) \rightarrow \{1, 2, \cdots, k\}$ such that $f(u) \neq f(v)$ for all $uv \in E(G)$. The color class S_i is the subset of vertices of G that is assigned to color i. The chromatic number $\chi(G)$ is the minimum number k for which G admits proper k - coloring. Equivalently the chromatic number $\chi(G)$ of a graph G is defined to be the minimum number of colors required to color the vertices of a graph G in such a way that no two adjacent vertices of G receive the same color. The minimum k such that we can partition $V(G) = S_1 \cup S_2 \cup \cdots \cup S_k$, where each S_i is independent set, is the chromatic number $\chi(G)$. A partition of $V(G)$ in to $\chi(G)$ independent sets is called χ - partition of G.

The set $D \subseteq V(G)$ of vertices in a graph G is called dominating set if every vertex $v \in V(G)$ is either an element of D or is adjacent to an element of D. The minimum cardinality of a dominating set is called the domination number of G which is denoted by $\gamma(G)$.

The domination in graph is one of the fastest growing concept in graph theory. Many variants of domination models are available in literature: independent domination, total domination, equitable domination, total equitable domination are among worth to mention. Independent sets play a significant role in graph theory in general. They appear in theory of trees, coloring of graphs and matching theory.

If $\mathcal{C} = \{S_1, S_2, \cdots, S_k\}$ is a k - coloring of a graph G then a subset D of $V(G)$ is called a transversal of $\mathcal{C}$ if $D \cap S_i \neq \phi$ for all $i \in \{1, 2, \cdots, k\}$. A dominating set D of a graph G is called a chromatic transversal dominating set (cdt - set) of G if D is transversal of every chromatic partition of G. The minimum cardinality of a cdt - set D of G is called the chromatic transversal domination number of G and is denoted by $\gamma_{ct}(G)$. This concept was introduced by Michaelraj et al. [4].

Definition 1.1 : The corona $G \circ H$ of two graphs G and H is defined to be the graph obtained by taking one copy of G of order n and n copies of H and joining i-th vertex of G with an edge to every vertex in the ith copy of H.

Definition 1.2 : The crown Cr_n is $C_n \circ K_1$ is obtained by joining a pendant edge to each vertex of C_n.

Definition 1.3 : The cartesian product of G and H is a graph, denoted as $G \times H$, whose vertex set $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent precisely if $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. Thus, $V(G \times H) = \{(g, h) = g \in V(G)$ and $h \in V(H)\}$ and $E(G \times H) = \{(g, h)(g', h.) = g = g', hh' \in E(H)$ or $gg' \in E(G), h = h'\}$.

Definition 1.4 : The book B_n is a graph $K_{1,n} \times P_2$.

Definition 1.5 : The square of a graph G denoted by G^2 has the same vertex set as of G and two vertices are adjacent in G^2 if they are at distance of 1 or 2 apart in G.

Definition 1.6 : The switching of a vertex v of G means removing all the edges incident to v and adding edges joining v to every vertex which is not adjacent to v in G. We denote the resultant graph by $\tilde{G}$.

In this paper we obtained the chromatic transversal domination number of some graphs.

Definition 1.7. [2] : For a bipartite graph G, $\gamma_{ct}(G) = \gamma(G)$ or $\gamma_{ct}(G) = \gamma(G) + 1$. All bipartite graphs for which $\gamma_{ct}(G) = \gamma(G)$ are called type I graphs. Other graphs are type II graphs.

Proposition 1.8 [2] : Let G and H be any graphs. Then

$$\gamma_{ct}(G \circ H) = \begin{cases} |V(G)|;. & \text{if } \ \chi(G) > \chi(H) \\ |V(G)| + \chi(H) - \chi(G); & \text{otherwise} \end{cases}.$$

Proposition 1.9 [8] : If $\tilde{C}_n$ is the graph obtained by switching of an arbitrary vertex v in cycle C_n then

$$\gamma(\tilde{C}_n) = \begin{cases} 2 & \text{if } \ n = 4 \\ 3; & \text{if } \ n > 4 \end{cases}$$

Proposition 1.10 [5] : If $\tilde{C}_n$ is the graph obtained by switching of an arbitrary vertex v in cycle C_n then

$$\chi(\tilde{C}_n) = \begin{cases} l & \text{if } \ n = 4 \\ 2; & \text{if } \ n = 5 \\ 3; & \text{if} n \geq 6 \end{cases}$$

Proposition 1.11 [3] : Let B_n be any book graph. Then $\gamma(B_n) = 2$, where $n \geq 3$.

Proposition 1.12 [6] : Let G be a graph with $\gamma(G) \leq \chi(G)$. Then $\gamma_{ct}(G) = \chi(G)$.

2. Main Results

Theorem 2.1 : Let G be a graph with $\chi(G) \leq 2$. Then $\gamma_{ct}(G) = \gamma(G)$, where G is not type II graph.

Proof : Let G be a graph with $\chi(G) \leq 2$ which is not type II. To prove the result, we consider following two cases:

Case I : If $\chi(G) = 1$ then G is null graph. Therefore $\gamma_{ct}(G) = \gamma(G)$.

Case II : If $\chi(G) = 2$ then G is bipartite graph. Moreover the graph G is not type II. Hence by Definition 1.7, $\gamma_{ct}(G) = \gamma(G)$. □

Theorem 2.2 : If G is a connected graph of order $n > 2$ with $\chi(G) > 2$ and G' is a graph obtained by duplication of every vertex of a connected graph G by an edge then $\gamma_{ct}(G') = n$.

Proof : If G is a connected graph of order $n > 2$ with $\chi(G) > 2$ and G' is a graph obtained by duplication of every vertex of a connected graph G by an edge. Then $G' \cong G \circ P_2$. We know that $\gamma_{ct}(G) > 2$ as $\chi(G) > 2$ and $\gamma_{ct}(P_2) = 2$. Hence by Proposition 1.8 $\gamma_{ct}(G') = n$ as $\chi(G) > \chi(P_2)$. □

Theorem 2.3: For the crown $Cr_n, \gamma_{ct}(Cr_n) = n$.

Proof : Let G be a cycle with n vertices and H be K_1. Then $Cr_n \cong C_n \circ K_1$. Moreover $\chi(C_n) = 2$ for n is even, $\chi(C_n) = 3$ for n is odd and $\chi(K_1) = 1$. Therefore $\chi(C_n) > \chi(K_1)$. Hence by Proposition 1.8, $\gamma_{ct}(Cr_n) = n$. □

Lemma 2.4 : For the book $B_n, \chi(B_n) = 2$, for all $n > 4$.

Proof : Let $B_n = K_{1,n} \times P_2$ be a book. Thus from the definition of B_n, B_n is a bipartite as B_n does not contains any odd cycle. Hence $\chi(B_n) = 2$. □

Theorem 2.5 : . For any book $B_n, \gamma_{ct}(B_n) = 2$, for all $n > 4$.

Proof : Let $B_n = K_{1,n} \times P_2$ be a book. Now by Lemma 2.4, $\chi(B_n) = 2$ and by Proposition 1.11, $\gamma(B_n) = 2$ for all $n > 4$. Therefore $\chi(B_n) = \gamma(B_n) = 2$. Hence, $\gamma_{ct}(B_n) = \chi(B_n) = 2$, for all $n > 4$. □

Lemma 2.6 :

$$\chi(C_n^2) = \begin{cases} 3; & \text{if } \ n \equiv 0 \ (mod 3) \\ 4; & \text{if } \ n \equiv 1 \ (mod \ 3) \\ 5; & \text{if } \ n \equiv 2 \ (mod \ 3) \end{cases}$$

Proof : Let $V(C_n) = V(C_n^2) = \{v_1, v_2, v_3, \cdots, v_n\}$ be the vertex set where $d_{C_n^2}(v_i) = 4$ for all $i \in \{1, 2, 3, \cdots, n\}$.

Moreover by definition of C_n^2, K_3 is a subgraph of C_{2n}. Therefore number of independent sets are atleast three. To prove the result we consider the following cases:

Case I : If $n \equiv 0$ $(mod\ 3)$.

Now we construct different sets of vertices ad follows:

$$\begin{aligned} S_1 &= \{v_{3i+1}/0 \leq i \leq \frac{n}{3} - 1\} \\ S_2 &= \{v_{3i+2}/0 \leq i \leq \frac{n}{3} - 1\} \\ S_3 &= \{v_{3i}/0 \leq i \leq \frac{n}{3}\} \end{aligned}$$

Further $V(C_n^2) = S_1 \cup S_2 \cup S_3$, where each S_i is a minimum independent set as K_3 is a subgraph of C_{2n}. Hence $\chi(C_n^2) = 3$ if $n \equiv 0$ $(mod\ 3)$.

Case II : If $n \equiv 1$ $(mod\ 3)$.

In this case we construct different sets of vertices ad follows:

$$\begin{aligned} S_1 &= \{v_{3i+1}/0 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor - 1\} \\ S_2 &= \{v_{3i+2}/0 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor - 1\} \\ S_3 &= \{v_{3i}/0 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor\} \\ S_4 &= \{v_n\} \end{aligned}$$

Further $V(C_n^2) = S1 \cup S_2 \cup S_3 \cup S_4$, where each S_i is a minimum independent set as K_3 is a subgraph of $C2_n$ and there exists a vertex $u \in S_i$ for all $i \in \{1, 2, 3\}$ such that $uv_n \in E(C_n^2)$. Hence $\chi(C_n^2) = 4$ if $n \equiv 1$ $(mod\ 3)$.

Case III : If $n \equiv 2\ (mod\ 3)$. Here we construct different sets of vertices as follows:

$$\begin{aligned} S_1 &= \{v_{3i+1}/0 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor - 1\} \\ S_2 &= \{v_{3i+2}/0 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor - 1\} \\ S_3 &= \{v_{3i}/0 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor\} \\ S_4 &= \{v_{n-1}\} \\ S_5 &= \{v_n\}. \end{aligned}$$

Further $V(C_n^2) = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$, where each S_i is a minimum independent set as K_3 is a subgraph of C_n^2 and there exists a vertex $u \in S_i$ for all $i \in \{1, 2, 3\}$ such that $uv_{n-1} \in E(C_n^2)$ and $uv_n \in E(C_n^2)$. Hence $\chi(C_n^2) = 5$ if $n \equiv 2\ (mod\ 3)$. □

Theorem 2.7 :

$$\gamma_{ct}(C_n^2) = \begin{cases} \lceil \frac{n}{5} \rceil; & \text{for } n \equiv 0\ (mod\ 3) \text{ and } n \geq 12 \\ \lceil \frac{n}{5} \rceil + 1; & \text{for } n \equiv 1\ (mod\ 3) \text{ and } n \geq 13 \\ \lceil \frac{n}{5} \rceil + 2; & \text{for } n \equiv 2\ (mod\ 3) \text{ and } n \geq 14 \end{cases}$$

Proof : If D is any color transversal dominating set of C_n^2 then without loss of generality $v_1 \in D$ as $d_{C_n^2}(v_i) = 4$ for all $i = \{1, 2, , \cdots, n\}$. To prove the result we consider the following cases:

Case I : For $n \equiv 0\ (mod\ 3)$.

In this case from the Lemma 2.6,

$$\begin{aligned} S_1 &= \{v_{3i+1}/0 \leq i \leq \frac{n}{3} - 1\} \\ S_2 &= \{v_{3i+2}/0 \leq i \leq \frac{n}{3} - 1\} \\ S_3 &= \{v_{3i}/0 \leq i \leq \frac{n}{3}\} \end{aligned}$$

are three minimum number of independent sets of vertices with color 1, 2 and 3 respectively.

Now we construct a set $D = \{V_{5i+1}/0 \leq i \leq \lceil \frac{n}{5} \rceil\}$. Then $|D| = \lfloor \frac{n}{5} \rfloor$. Moreover D is a chromatic transversal dominating set of C_n^2 as $D \cap S_i \neq \phi$ for all i. Further we claim that $|D|$ is a minimum because for any $u \in D, D - \{u\}$ is not a color transversal dominating set of C_n^2 as $(D - \{u\}) \cap S_i = \phi$ for some i. Therefore containing the vertices less than that of $|D|$ can not be a chromatic transversal dominating set of C_n^2. Hence $\gamma_{ct}(C_n^2) = \lceil \frac{n}{5} \rceil$.

Case II : For $n \equiv 1$ $(mod\ 3)$.

In this case from the Lemma 2.6,

$$\begin{aligned} S_1 &= \{v_{3i+1}/0 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor - 1\} \\ S_2 &= \{v_{3i+2}/0 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor - 1\} \\ S_3 &= \{v_{3i}/0 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor\} \\ S_4 &= \{v_n\} \end{aligned}$$

are minimum number of independent sets of vertices with color 1, 2, 3 and 4 respectively. Now we construct a set $D = \{V_{5i+1}/0i \leq \lfloor \frac{n}{5} \rfloor\} \cup \{v_n\}$. Then $|D| = \lceil \frac{n}{5} \rceil + 1$. Moreover D is a chromatic transversal dominating set of C_n^2 as $D \cap S_i \neq \phi$ or all i. Further we claim that $|D|$ is a minimum because for any $u \in D, D - \{u\}$ is not a color transversal dominating set of C_n^2 as $(D - \{u\}) \cap S_i = \phi$ for some i. Therefore containing the vertices less than that of $|D|$ can not be a chromatic transversal dominating set of C_n^2. Hence $\gamma_{ct}(C_n^2) = \lceil \frac{n}{5} \rceil + 1$.

Case III : For $n \equiv 1$ $(mod\ 3)$.

For case from the Lemma 2.6,

$$\begin{aligned} S_1 &= \{v_{3i+1}/0 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor - 1\} \\ S_2 &= \{v_{3i+2}/0 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor - 1\} \\ S_3 &= \{v_{3i}/0 \leq i \leq \left\lfloor \frac{n}{3} \right\rfloor\} \\ S_4 &= \{v_{n-1}\} \\ S_5 &= \{v_n\} \end{aligned}$$

are minimum number of independent sets of vertices with color 1, 2, 3, 4 and 5 respectively.

Now we construct a set $D = \{V_{5i+1}/0 \leq i \leq \lfloor \frac{n}{5} \rfloor\} \cup \{v_{n-1}, v_n\}$. Then $|D| = \lfloor \frac{n}{5} \rfloor + 2$. Moreover D is a chromatic transversal dominating set of C_n^2 as $D \cap S_i \neq \phi$ for all i. Further we claim that $|D|$ is a minimum because for any $u \in D, D - \{u\}$ is not a color transversal dominating set of C_n^2 as $(D - \{u\}) \cap S_i = \phi$ for some i. Therefore containing the vertices less than that of $|D|$ can not be a chromatic transversal dominating set of C_n^2. Hence $\gamma_{ct}(C_n^2) = \lceil \frac{n}{5} \rceil + 2$. □

Theorem 2.8 : Let $\tilde{C_n}$ be the graph obtained by switching of an arbitrary vertex v in cycle C_n then

$$\gamma_{ct}(\tilde{C_n}) = \begin{cases} 2; & \text{if } \ n = 4 \\ \\ 3;. & \text{if } \ n > 4 \end{cases}$$

Proof : Let $\tilde{C_n}$ be the graph obtained by switching of an arbitrary vertex v in cycle C_n. Now by Proposition 1.10 and 1.9, $\gamma(\tilde{C_n}) \leq \chi(\tilde{C_n})$. Therefore $\gamma_{ct}(\tilde{C_n}) = \chi(\tilde{C_n})$. Hence

$$\gamma_{ct}(\tilde{C_n}) = \begin{cases} 2; & \text{if } \ n = 4 \\ \\ 3;. & \text{if } \ n > 4 \end{cases}$$

□

3. Concluding Remarks

The concept of chromatic transversal dominating set is interesting as it relates two important concepts of graph theory, namely graph coloring and domination in graph. We have investigated this parameter for crown, book graph, C_n^2 and $\tilde{C_n}$. It has been shown that the graphs for which chromatic transversal domination number is equal to domination number.

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