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About m-domination number of graphs

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Abstract
In this paper, we have defined the concept of m-dominating set in graphs. In order to define this concept we have used the notion of m-adjacent vertices. We have also defined the concepts of minimal m-dominating set, minimum m-dominating set and m-domination number which is the minimum cardinality of an m-dominating set. We prove that the complement of a minimal m-dominating set is an m-dominating set. Also we prove a necessary and sufficient condition under which the m-domination number increases or decreases when a vertex is removed from the graph. Further we have also studied the concept of m-removing a vertex from the graph and we prove that the m-removal of a vertex from the graph always increases or does not change the m-domination number. Some examples have also been given.

Keywords
m-dominating set, minimal m-dominating set, minimum m-dominating set, private m-neighbourhood of a vertex, m-removal of a vertex.

AMS Subject Classification
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Contents

1. Introduction

In the area of mixed domination several new concepts have been appeared. The concept of a vertex which m-dominates an edge and the concept of an edge which m-dominates a vertex have been defined and studied by some authors like R. Laskar, K. Peters, E. Sampathkumar, S. S. Kamath and others [3–5]. The above concepts can be used to define m-adjacent vertices and m-adjacent edges. In fact, we have defined m-adjacent vertices and m-adjacent edges in [1]. We observe that these concepts give rise to new concept called m-dominating set using m-adjacent vertices.

We also introduce the concepts of minimal m-dominating set, minimum m-dominating set and m-domination number which is the minimum cardinality of an m-dominating set.

We have also a concept called m-removal of a vertex in graphs which has been introduced in [2]. We proved the effect of m-removing a vertex on m-domination number.

2. Preliminaries and Notations

If G is a graph then $E(G)$ denotes the edge set and $V(G)$ denotes the vertex set of the graph. If v is a vertex of G then $G\backslash v$ denotes the subgraph of G obtained by removing the vertex v and all the edges incident to v. $N(v)$ denotes the set of vertices which are adjacent to v. $N[v] = N(v) \cup v$. If x is any vertex then $d(x)$ denotes the degree of x and is the number of edges incident at x.

Definition 2.1. *[1] Let u and v be two vertices of G. Then u and v are said to be m-adjacent vertices in G if there is an edge of G which m-dominates both u and v in G.*

Definition 2.2. *[2] Let G be a graph and $v \in V(G)$. We obtain a subgraph of G by removing vertex v and certain edges which is called the subgraph obtained by m-removing the vertex v from the graph G.*

Definition 2.3. *[2] Let G be a graph and $v \in V(G)$. The subgraph obtained by m-removing vertex v from G has the vertex set $V(G)\backslash\{v\}$ and by removing all the edges of G which m-dominate vertex v. This subgraph is denoted as $G\backslash^m\{v\}$.*

3. Main Results

Definition 3.1. *Let G be a graph and $S \subset V(G)$. Then S is said to be an m-dominating set if for every vertex v in $V(G) \setminus S$, there is a vertex u in S such that u and v are m-adjacent.*

Note that every dominating set is an m-dominating set but m-dominating set need not be a dominating set.

Example 3.2. *Consider the path graph P_5 with vertices $\{v_1, v_2, v_3, v_4, v_5\}$*

Figure 1. P_5

Let $S = \{v_3\}$ then S is an m-dominating set but not dominating set.

Definition 3.3. *Let G be a graph and $S \subset V(G)$ be an m-dominating set. Then S is said to be a minimal m-dominating set if $S\setminus\{v\}$ is not an m-dominating set for every v in S.*

Definition 3.4. *An m-dominating set with minimum cardinality is called a minimum m-dominating set. The cardinality of minimum m-dominating set is the m-domination number of the graph G and it is denoted as $\gamma_{mv}(G)$.*

Definition 3.5. *Let G be a graph and $v \in V(G)$. Then v is said to be an m-isolated vertex of G if for every other vertex u of G, u is not m-adjacent to v.*

Obviously, a vertex v is isolated if and only if it is m-isolated.

Theorem 3.6. *Let G be a graph and $S \subset V(G)$ be an m-dominating set of G. Then S is a minimal m- dominating set of G if and only if for every $u \in S$ atleast one of the following two conditions holds.*

(i) u is not m-adjacent to any other vertex of S.

(ii) There exist a vertex $v \in V(G) \setminus S$ such that v is m-adjacent to only one vertex of S namely u.

Proof. Suppose S is a minimal m-dominating set. Let $u \in S$. Now $S\setminus\{u\}$ is not an m-dominating set. Therefore, there is a vertex v outside $S\setminus\{u\}$ such that v is not m-adjacent to any vertex of $S\setminus\{u\}$.
Case (i): $v = u$
Then u is not m-adjacent to any other vertex of S.
Case (ii): $v \neq u$
Then $v \notin S$.
Subcase (i): v is not m-adjacent to any vertex of $S\setminus\{u\}$.
Subcase (ii): v is m-adjacent to some vertex of S.
Therefore, v is m-adjacent to only one vertex of S namely u.
Conversely, suppose any of condition (i) and (ii) is satisfied for any $u \in S$.
Let $u \in S$.
Case (i): Suppose condition (i) is satisfied.
Therefore, u is not m-adjacent to any vertex of $S\setminus\{u\}$ and also $u \notin S\setminus\{u\}$.
Case (ii): Suppose condition (ii) is satisfied.
Let $v \in V(G) \setminus S$ such that v is m-adjacent to only one vertex of S namely u. Then v is not m-adjacent to any vertex of $S\setminus\{u\}$. Thus it follows that $S\setminus\{u\}$ is not an m-dominating set of G for any $u \in S$.
Therefore, S is a minimal m-dominating set. □

Theorem 3.7. *Let G be a graph without m-isolated vertices and S be a minimal m-dominating set of G. Then $V(G) \setminus S$ is an m-dominating set of G.*

Proof. Let $v \in S$. Since S is a minimal m-dominating set, (i) or (ii) of theorem (3.6) is satisfied.
Suppose (i) is satisfied. Then v is not m-adjacent with any other vertex of S. Since v is not an m-isolated vertex of G, v is m-adjacent to some vertex u of G. Then $u \in V(G) \setminus S$.
Suppose (ii) is satisfied and suppose v is m-adjacent to some vertex of S. Now, there is a vertex u in $V(G) \setminus S$ such that u is m-adjacent to v and u is not m-adjacent to any other vertex of S.
Thus in both the cases v is m-adjacent to some vertex of $V(G) \setminus S$. Therefore, $V(G) \setminus S$ is an m-dominating set of G. □

Corollary 3.8. *Let G be a graph without m-isolated vertices. Then $\gamma_{mv}(G) \leq n/2$.*

Proof. Let S be a minimum m-dominating set of G. Then $\gamma_{mv}(G) = |S|$. Now by the theorem(3.7), $V(G) \setminus S$ is also an m-dominating set. Therefore, $\gamma_{mv}(G) \leq |V(G) \setminus S|$. Therefore, $\gamma_{mv}(G) = min\{|S|, |V(G) \setminus S|\}$. If $|S| \leq n/2$ then $\gamma_{mv}(G) \leq n/2$. If $|V(G) \setminus S| > n/2$ then $|S| < n/2$ and therefore $\gamma_{mv}(G) \leq n/2$. □

Definition 3.9. *Let G be a graph and $x \in V(G)$. The m-vertex open neighbourhood of x (or simply m-open neighbourhood of x) is the set $N_{mv}(x) = \{u \in V(G)$ such that u is m-adjacent to $x\}$.*
Also the m-vertex closed neighbourhood of x is the set $N_{mv}[x] = N_{mv}(x) \cup \{x\}$.

Now we state and prove a necessary and sufficient condition under which the m-domination number of a graph increases when a vertex is removed from the graph.

Theorem 3.10. *Let G be a graph and $v \in V(G)$. Then $\gamma_{mv}(G\setminus v) > \gamma_{mv}(G)$ if and only if following conditions are satisfied*

(i) v is not an m-isolated vertex of G.

(*ii*) *If S is a minimum m-dominating set of G and $v \notin S$ then there is a vertex x in $V(G) \backslash S$ such that $x \neq v$ and $d(x,S) > 3$ in the subgraph $G \backslash v$.*

(*iii*) *There is no subset S of $V(G) \backslash N_{mv}[v]$ such that $|S| \leq \gamma_{mv}(G)$ and it is an m-dominating set of $G \backslash v$.*

Proof. Suppose $\gamma_{mv}(G \backslash v) > \gamma_{mv}(G)$.

(*i*) Suppose v is an m-isolated vertex of G. Let S be any minimum m-dominating set of G. Then $v \in S$. Let $S_1 = S \backslash \{v\}$. Let x be any vertex of $G \backslash v$ such that $x \notin S_1$. Then, $x \notin S$. Since S is an m-dominating set of G, $d(x,S) \leq 3$ in G. Now v is an m-isolated vertex, $d(x,S_1)$ in $G = d(x,S_1)$ in $G \backslash v$. Therefore, $d(x,S_1)$ in $G \backslash v \leq 3$. Thus, x is m-adjacent to some member of S_1 in $G \backslash v$. This proves that S_1 is an m-dominating set in $G \backslash v$. Therefore $\gamma_{mv}(G \backslash v) \leq |S_1| < |S| = \gamma_{mv}(G)$, which is a contradiction. Therefore, v cannot be an m-isolated vertex of G.

(*ii*) Suppose, there is a minimum m-dominating set S of G such that $v \notin S$. Suppose for every vertex x which is not in S and $x \neq v$, $d(x,S) \leq 3$ in $G \backslash v$. Then S is an m-dominating set in $G \backslash v$. This implies that $\gamma_{mv}(G \backslash v) \leq |S| = \gamma_{mv}(G)$ which is a contradiction. Therefore (*ii*) is satisfied.

(*iii*) Suppose, there is a subset S of $V(G) \backslash N_{mv}[v]$ such that $|S| \leq \gamma_{mv}(G)$ and S is an m-dominating set of $G \backslash v$. Then $\gamma_{mv}(G \backslash v) \leq |S| \leq \gamma_{mv}(G)$ which is again a contradiction. Therefore, (*iii*) holds.

Conversely, suppose condition (*i*), (*ii*) and (*iii*) are satisfied. First suppose that $\gamma_{mv}(G \backslash v) = \gamma_{mv}(G)$. Let S be a minimum m-dominating set of $G \backslash v$. Let x be any vertex of G such that $x \notin S$ and $x \neq v$. Then $d(x,S)$ in $G \leq d(x,S)$ in $G \backslash v$ which is ≤ 3. Now suppose v is m-adjacent to some vertex of S. Then S is a minimum m-dominating set of G and $v \notin S$. If $x \in V(G) \backslash S$ such that $x \neq v$ then $d(x,S) \leq 3$ in $G \backslash v$. This contradicts condition (*ii*). Therefore, v cannot be an m-adjacent to any vertex of S. Then S is a subset of $V(G) \backslash N_{mv}[v]$. Also, $|S| \leq \gamma_{mv}(G)$. Also, S is an m-dominating set of $G \backslash v$. This contradicts condition (*iii*). Thus, $\gamma_{mv}(G \backslash v) = \gamma_{mv}(G)$ is not possible.
Suppose, $\gamma_{mv}(G \backslash v) < \gamma_{mv}(G)$.
Let S be a minimum m-dominating set of $G \backslash v$. Since $|S| < \gamma_{mv}(G)$, S cannot be an m-dominating set of G. Therefore, v cannot be m-adjacent to any vertex of G. Therefore, S is a subset of $V(G) \backslash N_{mv}[v]$. Also $|S| \leq \gamma_{mv}(G)$. Also S is an m-dominating set of $G \backslash v$. This again contradicts (*iii*). Therefore, $\gamma_{mv}(G \backslash v) < \gamma_{mv}(G)$ is also not possible. Thus, $\gamma_{mv}(G \backslash v) > \gamma_{mv}(G)$. □

Corollary 3.11. *Let G be a graph and $v \in V(G)$ be such that $\gamma_{mv}(G \backslash v) > \gamma_{mv}(G)$ then $d(v,S) \leq 2$ for every minimum m-dominating set S of G.*

Proof. Let S be any minimum m-dominating set of G. Suppose $v \notin S$. By (*ii*) of theorem(3.10), there is a vertex x in $V(G) \backslash S$ such that $d(x,S) > 3$ in $G \backslash v$. However, $d(x,S) \leq 3$ in G. Therefore, there is a vertex y in S such that $d(x,y) \leq 3$. Any path from x to y in G must contain v as an internal vertex (otherwise v does not appear in the path and therefore there is a path of length less than or equal to 3 between x and y in $G \backslash v$). Obviously, there is a path from v to y of length ≤ 2. Therefore, $d(v,S) \leq 2$. . □

Definition 3.12. *Let G be a graph, $v \in V(G)$ and $S \subset V(G)$ such that $v \in S$. Then private m-neighbourhood of v with respect to S is defined as $P_{mn}[v,S] = \{u \in V(G)$ such that $N_{mv}[u] \cap S = \{v\}\}$.*

Remark 3.13. *Note that if $v \in S$ and v is not m-adjacent to any other vertex of S then $v \in P_{mn}[v,S]$. If $u \in V(G) \backslash S$ then $u \in P_{mn}[v,S]$ if and only if u is m-adjacent to only one vertex of S namely v.*

Now we state and prove a necessary and sufficient condition under which the m-domination number of a graph decreases when a vertex is removed from the graph.

Theorem 3.14. *Let G be a graph and $v \in V(G)$. Then $\gamma_{mv}(G \backslash v) < \gamma_{mv}(G)$ if and only if there is a minimum m-dominating set S of G such that $v \in S$ and $P_{mn}[v,S] = \{v\}$.*

Proof. Suppose $\gamma_{mv}(G \backslash v) < \gamma_{mv}(G)$. Let S_1 be a minimum m-dominating set of $G \backslash v$. Then S_1 cannot be an m-dominating set of G. Therefore, $d(v,S_1) > 3$. Let $S = S_1 \cup \{v\}$. Let $x \in V(G) \backslash S$ then $x \notin S_1$. Since S_1 is an m-dominating set of $G \backslash v$, x is m-adjacent to some vertex z of S_1 in $G \backslash v$. Then x is m-adjacent to z in G also. Thus S is an m-dominating set of G and $v \in S$. Note that as mentioned above v is not m-adjacent to any other vertex of S in G. Therefore, $v \in P_{mn}[v,S]$. Let $x \in V(G) \backslash S$ such that x is m-adjacent to v in G. Now, x is m-adjacent to y in S(in $G \backslash v$) such that $y \neq v$. Then x is also m-adjacent to y in G. Thus x is m-adjacent to two distinct vertices of S. Therefore, $x \notin P_{mn}[v,S]$ if $x \in V(G) \backslash S$. Thus $P_{mn}[v,S] = \{v\}$.
Conversely, suppose there is a minimum m-dominating set S of G such that $v \in S$ and $P_{mn}[v,S] = \{v\}$. Let $S_1 = S \backslash \{v\}$. Let x be a vertex of $G \backslash v$ such that $x \notin S_1$. Then $x \notin S$. Since S is an m-dominating set of G, x is m-adjacent to some vertex y of S. Suppose $y = v$. Now $x \notin P_{mn}[v,S]$. Therefore, x is m-adjacent to some vertex z of S in G such that $z \neq v$. Therefore, $d(x,z) \leq 3$ in G. Let P be a path in G joining x to z. If v is a vertex in this path then it will imply that $d(v,z) \leq 3$ and this implies that v is m-adjacent to z and $z \in S$. This contradicts the fact that $v \in P_{mn}[v,S]$. Thus, v does not appear in this path. Thus P is a path in $G \backslash v$ joining x to z. Therefore, x is m-adjacent to z in $G \backslash v$ and $z \in S_1$. Thus S_1 is an m-dominating set in $G \backslash v$. Thus, $\gamma_{mv}(G \backslash v) \leq |S_1| < |S| = \gamma_{mv}(G)$. □

Corollary 3.15. *Let G be a graph and $v \in V(G)$ be such that v is not m-isolated vertex of G. If $\gamma_{mv}(G \backslash v) < \gamma_{mv}(G)$ then there is a minimum m-dominating set S such that $v \notin S$.*

Proof. There is a minimum m-dominating set S_1 of G such that $v \in S_1$ and $P_{mn}[v, S_1] = \{v\}$. Since v is not an m-isolated vertex in G, there is a vertex x which is m-adjacent to v in G. Since v is not m-adjacent to any vertex of S_1, $x \in V(G) \setminus S_1$. Let $S = (S_1 \setminus \{v\}) \cup \{x\}$. Then $|S| = |S_1| = \gamma_{mv}(G)$. Also $v \notin S$. Let $z \in V(G) \setminus S$. If $z = v$ then z is m-adjacent to x and $x \in S$. Suppose $z \neq v$. Then $z \notin S_1$. Since S_1 is an m-dominating set of G, z is m-adjacent to some vertex t of S_1. If $t = v$ then z is m-adjacent to some vertex t' of S_1 such that $t' \neq v$ because $z \notin P_{mn}[v, S_1]$. Thus, z is m-adjacent to some vertex t' of S. Thus S is an m-dominating set of G. Thus, S is a minimum m-dominating set of G such that $v \notin S$. □

Theorem 3.16. *Let G be a graph and $v \in V(G)$ such that v is not an m-isolated vertex in G. Then $\gamma_{mv}(G \setminus v) < \gamma_{mv}(G)$ if and only if there is a minimum m-dominating set S not containing v and a vertex x in S such that $P_{mn}[x, S] = \{v\}$.*

Proof. Suppose $\gamma_{mv}(G \setminus v) < \gamma_{mv}(G)$. By theorem(3.14), there is a minimum m-dominating set S_1 such that $v \in S_1$ and $P_{mn}[v, S_1] = \{v\}$. Let x be a vertex in $V(G) \setminus S_1$, which is adjacent to v. Let $S = (S_1 \setminus \{v\}) \cup \{x\}$. Then $x \in S$ and by the corollary(3.15), S is a minimum m-dominating set of G not containing v. Note that v is not m-adjacent to any vertex of S_1 because $v \in P_{mn}[v, S_1]$. Therefore, v is adjacent to only one vertex of S namely x. Thus $v \in P_{mn}[x, S]$. Again x is m-adjacent to v and since $x \notin P_{mn}[v, S_1]$, x is m-adjacent to some vertex y of S_1 where $y \neq v$. Therefore, x is m-adjacent to some vertex of S and therefore $x \notin P_{mn}[x, S]$. Let z be a vertex of $V(G) \setminus S$ such that z is m-adjacent to x. Since $z \notin S_1$, z is m-adjacent to some vertex w of S_1 because S_1 is an m-dominating set of G. Thus, z is m-adjacent to two distinct vertices of S namely x and w. Therefore, $z \notin P_{mn}[x, S]$. Hence, $P_{mn}[x, S] = \{v\}$.
Conversely, suppose there is a minimum m-dominating set S such that $v \notin S$ and for some vertex x in S, $P_{mn}[x, S] = \{v\}$. Let $S_1 = S \setminus \{x\}$. Now, $x \notin P_{mn}[x, S]$. Therefore, x is m-adjacent to some vertex y of S in G. Note that v is not m-adjacent to any vertex of S except x. Let P be a path in G from x to y whose length is ≤ 3. If v is an internal vertex in this path then it would imply that $d(v, y) \leq 3$ in G and this means that v is m-adjacent to y in G and $y \neq x$. This is a contradiction. Thus v cannot appear as an internal vertex in the path above from x to y. Therefore, this is a path in $G \setminus v$ from x to y having length ≤ 3. Thus x is m-adjacent to y in $G \setminus v$ and $y \in S_1$. Let z be any vertex of $G \setminus v$ such that $z \notin S_1$ and $z \neq x$. Then $z \notin S$. Now, z is m-adjacent to some vertex w of S in G. If $w = x$ then there is another vertex w' in S such that z is m-adjacent to w' in G. By the same reasoning as given above we say that z is m-adjacent to w' in $G \setminus v$ also. Also $w' \in S_1$. Thus, we have proved that S_1 is an m-dominating set of $G \setminus v$. Therefore, $\gamma_{mv}(G \setminus v) \leq |S_1| < |S| = \gamma_{mv}(G)$. Hence, $\gamma_{mv}(G \setminus v) < \gamma_{mv}(G)$. □

Example 3.17. *Consider the path graph P_8 with vertices $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$*

Figure 2. P_8

Here, $\gamma_{mv}(G) = 2$ and $\gamma_{mv}(G \setminus \{v_8\}) = 1$. Let $S = \{v_4, v_5\}$. Then $P_{mn}[v_5, S] = \{v_8\}$

Corollary 3.18. *Let G be a graph and $v \in V(G)$ be such that $d(v, S) = 3$ for every minimum m-dominating set S of G. Then $\gamma_{mv}(G \setminus v) = \gamma_{mv}(G)$.*

Proof. If $\gamma_{mv}(G \setminus v) > \gamma_{mv}(G)$ then $d(v, S) \leq 2$ for every minimum m-dominating set S of G which is a contradiction. If $\gamma_{mv}(G \setminus v) < \gamma_{mv}(G)$ then there is a minimum m-dominating set S of G such that $d(v, S) = 0$ which is again a contradiction. Therefore, $\gamma_{mv}(G \setminus v) = \gamma_{mv}(G)$. □

Proposition 3.19. *Let G be a graph and F be a set of edges of G. Then $\gamma_{mv}(G \setminus F) \geq \gamma_{mv}(G)$.*

Proof. Let S be a minimum m-dominating set of $G \setminus F$. Let $x \in V(G) \setminus S$. Now, x is m-adjacent to some vertex y of S in $G \setminus F$. Therefore, there is an edge e in the graph $G \setminus F$ which m-dominates both x and y. Therefore, e m-dominates x and y in G also. Therefore, x and y are m-adjacent in G also. Thus, x is m-adjacent to some vertex y of S in G. Therefore, $\gamma_{mv}(G \setminus F) \geq |S| = \gamma_{mv}(G)$. □

Proposition 3.20. *Let G be a graph and $v \in V(G)$. Then, $\gamma_{mv}(G \setminus^m \{v\}) \geq \gamma_{mv}(G \setminus v)$.*

Proof. Note that $G \setminus^m \{v\}$ is obtained by removing those edges of G which m-dominate v but which are not incident to v. These are the edges of $G \setminus v$. Let F be the set of these edges. Then by the proposition(3.19), $\gamma_{mv}(G \setminus^m \{v\}) = \gamma_{mv}((G \setminus v) \setminus F) \geq \gamma_{mv}(G \setminus v)$. □

Proposition 3.21. *Let G be a graph and $v \in V(G)$ be a non-isolated vertex of G. Then $\gamma_{mv}(G \setminus^m \{v\}) \geq \gamma_{mv}(G)$.*

Proof. Let T be a minimum m-dominating set of $G \setminus^m \{v\}$. Then T contains all m-isolated vertices of $G \setminus^m \{v\}$. Now every neighbour of v is an m-isolated vertex of $G \setminus^m \{v\}$. Therefore, every neighbour of v is an element of T. Thus T is an m-dominating set of G. Therefore, $\gamma_{mv}(G) \leq |T| = \gamma_{mv}(G \setminus^m \{v\})$. □

Theorem 3.22. *Let G be a graph and $v \in V(G)$ be such that $d(v) \geq 2$. Then $\gamma_{mv}(G \setminus^m \{v\}) > \gamma_{mv}(G)$.*

Proof. Suppose S is a minimum m-dominating set of $G \setminus^m \{v\}$. Let $S_1 = (S \setminus N(v)) \cup \{v\}$. Then $|S_1| < |S|$. Let x be any vertex of G such that $x \notin S_1$. If $x \in N(v)$ then x is adjacent to v and of course $v \in S_1$. Suppose, $x \notin N(v)$. Then $x \notin S$ and also $x \neq v$. Thus x is a vertex of $G \setminus^m \{v\}$ and $x \notin S$. Therefore, x is m-adjacent to some vertex y of S. Therefore, $d(x, y) \leq 3$

in $G\backslash^m\{v\}$. Since elements of $N(v)$ are isolated vertices in $G\backslash^m\{v\}$, $y \notin N(v)$ and hence $y \in S_1$. Also $d(x,y) \leq 3$ in G. Thus, x is m-adjacent to y where $y \in S_1$. Thus, S_1 is an m-dominating set in G. Therefore, $\gamma_{mv}(G) \leq |S_1| < |S| = \gamma_{mv}(G\backslash^m\{v\})$. □

Definition 3.23. *Let G be a graph, $S \subset V(G)$ and $v \in S$. Then the external private m-neighbourhood of v with respect to S is $E_xP_{m,n}[v,S] = \{w \in V(G) \backslash S$ such that w is m-adjacent to v in G but w is not m-adjacent to any other member of $S\}$.*

Theorem 3.24. *Let G be a graph. v be a pendant vertex of G and u be its neighbour. Then $\gamma_{mv}(G\backslash^m\{v\}) = \gamma_{mv}(G)$ if and only if there is a minimum m-dominating set S of G such that $u \in S$, $v \notin S$ and $E_xP_{m,n}[u,S] \subseteq \{v\}$.*

Proof. It is already true that $\gamma_{mv}(G\backslash^m\{v\}) \geq \gamma_{mv}(G)$. Suppose there is a minimum m-dominating set S of G such that $u \in S$, $v \notin S$ and the condition is satisfied. Let x be a vertex of $G\backslash^m\{v\}$ such that $x \notin S$. Now x is m-adjacent to some vertex y of S in G. If $y = u$ then x is not m-adjacent to u in $G\backslash^m\{v\}$. Since the condition is satisfied, x is m-adjacent in $G\backslash^m\{v\}$ to some vertex z of S such that $z \neq u$. If x is not m-adjacent to u then x is m-adjacent in G to some vertex w in S such that $w \neq u$. Then x is m-adjacent to w in $G\backslash^m\{v\}$ also ($\because$ The path joining x and w cannot contain u as x is not m-adjacent to u). Thus from both the above cases it follows that S is an m-dominating set in $G\backslash^m\{v\}$. Thus, $\gamma_{mv}(G\backslash^m\{v\}) \leq |S| = \gamma_{mv}(G)$. Hence, $\gamma_{mv}(G\backslash^m\{v\}) = \gamma_{mv}(G)$.

Conversely, suppose $\gamma_{mv}(G\backslash^m\{v\}) = \gamma_{mv}(G)$. Let S be a minimum m-dominating set of $G\backslash^m\{v\}$. Since u is an isolated vertex in $G\backslash^m\{v\}$, $u \in S$. Obviously, $v \notin S$. Let z be a vertex such that $z \notin S$ and $z \neq v$. Suppose, z is m-adjacent to u in G. Since S is an m-dominating set of $G\backslash^m\{v\}$, z is m-adjacent in $G\backslash^m\{v\}$ to some vertex u' of S. Note that $u' \neq u$ because u is an isolated vertex in $G\backslash^m\{v\}$. Now $d(z,u') \leq 3$ in $G\backslash^m\{v\}$. Therefore, $d(z,u') \leq 3$ in G. Thus we have proved that $z \in V(G)\backslash S$, $z \neq v$ and if z is m-adjacent to u in G then z is also m-adjacent to some other vertex u' of S in $G\backslash^m\{v\}$. Note that S is an m-dominating set in G also. Since $\gamma_{mv}(G\backslash^m\{v\}) = \gamma_{mv}(G)$, S is a minimum m-dominating set of G and the condition is satisfied. □

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