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On Kasaj topological spaces

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Abstract
In year 2013, L. Thivagar et al. introduced nano topological space and he analysed some properties of weak open sets. In this paper we shall introduce Kasaj-topological space. We shall introduce some new classes of weak open sets namely Kasaj-pre-open sets and Kasaj-semi-open sets in Kasaj topological spaces and analyze their basic properties. We shall also define new types of continuous functions namely Kasaj-continuous function, Kasaj-pre-continuous function, Kasaj-semi-continuous function in Kasaj topological space.

Keywords
Kasaj topological space, Kasaj-pre-open set, Kasaj-semi-open set.

AMS Subject Classification
54A05, 54B05.

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Contents

1. Introduction

In recent times many people have introduced new topological space and it is studied very well. For example, nano topological space was introduced by L. Thivagar et al. [3]. S. Chandrasekar [5] introduced Micro topological spaces which are extension of nano topological spaces. He has used Levine's simple extension concepts in nano topological spaces. The notations of Semi-open sets and Pre-open sets were introduced by Levine [4], Mashhour et al. [1], respectively. In this paper, we shall define new topological space namely Kasaj topological spcae. We shall also define Kasaj-pre-open set and Kasaj-semi-open set, investigate basic properties and find the relation between these new classes. We shall also define new types of continuous function namely Kasaj-continuous function, Kasaj-pre-continuous function, Kasaj-semi-continuous function.

2. Preliminary

Definition 2.1. *A subset $\mathfrak{P}$ of a topological space $(\mathfrak{X},\mathfrak{I})$ is called*

- *a semi-open set [4] if $\mathfrak{P} \subseteq cl(int(\mathfrak{P}))$.*
- *a pre-open set [1] if $\mathfrak{P} \subseteq int(cl(\mathfrak{P}))$.*

The complement of a semi-open set (pre-open set) in a space $\mathfrak{X}$ is called semi-closed set (pre-closed set) in $\mathfrak{X}$.

2.1 Nano Topological Spaces

Definition 2.2. *[3] Let $\mathfrak{A}$ be a non-empty Universal set and $\mathfrak{R}$ be an equivalence relation on $\mathfrak{A}$ and it is named as* the indiscernibility relation. *The pair $(\mathfrak{A},\mathfrak{R})$ is called as* approximation space. *Let $\mathfrak{X} \subseteq \mathfrak{A}$.*

1. *The lower approximation of $\mathfrak{X}$ with respect to $\mathfrak{R}$ is denoted by $\mathscr{L}_{\mathfrak{R}}(\mathfrak{X})$ and is defined by*

$$\mathscr{L}_{\mathfrak{R}}(\mathfrak{X}) = \cup_{x\in\mathfrak{A}} \{P(x) : P(x) \subseteq \mathfrak{X}\}$$

where $P(x)$ denotes the equivalence relation which contains $x \in \mathfrak{A}$.

2. *The upper approximation of $\mathfrak{X}$ with respect to $\mathfrak{R}$ is denoted by $\mathscr{U}_{\mathfrak{R}}(\mathfrak{X})$ and is defined by*

$$\mathscr{U}_{\mathfrak{R}}(\mathfrak{X}) = \cup_{x\in\mathfrak{A}} \{P(x) : P(x) \cap \mathfrak{X} \neq \emptyset\}$$

where $P(x)$ denotes the equivalence relation which contains $x \in \mathfrak{A}$.

3. *The boundary region of $\mathfrak{X}$ with respect to $\mathfrak{R}$ is denoted by $\mathfrak{Q}_{\mathfrak{R}}(\mathfrak{X})$ and is defined by*

$$\mathfrak{Q}_{\mathfrak{R}}(\mathfrak{X}) = \mathscr{U}_{\mathfrak{R}}(\mathfrak{X}) \setminus \mathscr{L}_{\mathfrak{R}}(\mathfrak{X}).$$

Definition 2.3. *[3] Let $\mathfrak{A}$ be an universal set. $\mathfrak{R}$ be an equivalence relation on $\mathfrak{A}$, $\mathfrak{X} \subseteq \mathfrak{A}$ and $\mathfrak{I}_{\mathfrak{R}}(\mathfrak{X}) = \{\mathfrak{A}, \emptyset, \mathscr{L}_{\mathfrak{R}}(\mathfrak{X}), \mathscr{U}_{\mathfrak{R}}(\mathfrak{X}), \mathfrak{Q}_{\mathfrak{R}}(\mathfrak{X})\}$ which satisfies the following axioms.*

1. *$\mathfrak{A}, \emptyset \in \mathfrak{I}_{\mathfrak{R}}(\mathfrak{X})$.*
2. *The union of elements of any subcollection of $\mathfrak{I}_{\mathfrak{R}}(\mathfrak{X})$ is in $\mathfrak{I}_{\mathfrak{R}}(\mathfrak{X})$.*
3. *The intersection of any finite subcollection of elements of $\mathfrak{I}_{\mathfrak{R}}(\mathfrak{X})$ is in $\mathfrak{I}_{\mathfrak{R}}(\mathfrak{X})$.*

Then $(\mathfrak{A}, \mathfrak{I}_{\mathfrak{R}}(\mathfrak{X}))$ *is called* nano topological space. *The members of $\mathfrak{I}_{\mathfrak{R}}(\mathfrak{X})$ are called* nano open sets.

3. Kasaj Topological Space

Definition 3.1. *Let $(\mathfrak{A}, \mathfrak{I}_{\mathfrak{R}}(\mathfrak{X}))$ be a nano topological space and Kasaj topology is defined by $KS_{\mathfrak{R}}(\mathfrak{X}) = \{(K \cap S) \cup (K' \cap S') : K, K' \in \mathfrak{I}_{\mathfrak{R}}(\mathfrak{X})$, fixed $S, S' \notin \mathfrak{I}_{\mathfrak{R}}(\mathfrak{X}), S \cup S' = \mathfrak{A}\}$ and is called* Kasaj topological space.

Definition 3.2. *The Kasaj topology $KS_{\mathfrak{R}}(\mathfrak{X})$ satisfies the following postulates :*

1. *$\mathfrak{A}, \emptyset \in KS_{\mathfrak{R}}(\mathfrak{X})$.*
2. *The union of elements of any subcollection of $KS_{\mathfrak{R}}(\mathfrak{X})$ is in $KS_{\mathfrak{R}}(\mathfrak{X})$.*
3. *The intersection of any finite subcollection of elements of $KS_{\mathfrak{R}}(\mathfrak{X})$ is in $KS_{\mathfrak{R}}(\mathfrak{X})$.*

Then $(\mathfrak{A}, \mathfrak{I}_{\mathfrak{R}}(\mathfrak{X}), KS_{\mathfrak{R}}(\mathfrak{X}))$ is called Kasaj topological spaces and the members of $KS_{\mathfrak{R}}(\mathfrak{X})$ are called Kasaj open sets (KS-open sets) *and the complement of a Kasaj-open set is called a Kasaj-closed(KS-closed) set and the collection of all Kasaj-closed sets is denoted by $KSCL(\mathfrak{X})$.*

Definition 3.3. *The Kasaj closure and the Kasaj interior of a set $\mathfrak{P}$ is denoted by $KS_{cl}(\mathfrak{P})$ and $KS_{int}(\mathfrak{P})$, respectively. It is defined by*

$$KS_{cl}(\mathfrak{P}) = \cap\{\mathfrak{Q} : \mathfrak{P} \subseteq \mathfrak{Q}, \mathfrak{Q} \text{ is } KS-closed\}$$

$$KS_{int}(\mathfrak{P}) = \cup\{\mathfrak{Q} : \mathfrak{Q} \subseteq \mathfrak{P}, \mathfrak{Q} \text{ is } KS-open\}.$$

Remark 3.4.

1. *$KS_{int}(\mathfrak{P})$ is the largest KS-open set contained in $\mathfrak{P}$.*
2. *$KS_{cl}(\mathfrak{P})$ is the smallest KS-closed set containing $\mathfrak{P}$.*

Definition 3.5. *For any two subsets $\mathfrak{P}, \mathfrak{Q}$ of $\mathfrak{A}$ in a Kasaj topological space $(\mathfrak{A}, \mathfrak{I}_{\mathfrak{R}}(\mathfrak{X}), KS_{\mathfrak{R}}(\mathfrak{X}))$,*

1. *$\mathfrak{P}$ is a Kasaj-closed set if and only if $KS_{cl}(\mathfrak{P}) = \mathfrak{P}$.*
2. *$\mathfrak{P}$ is a Kasaj-open set if and only if $KS_{int}(\mathfrak{P}) = \mathfrak{P}$.*
3. *If $\mathfrak{P} \subseteq \mathfrak{Q}$, then $KS_{int}(\mathfrak{P}) \subseteq KS_{int}(\mathfrak{Q})$ and $KS_{cl}(\mathfrak{P}) \subseteq KS_{cl}(\mathfrak{Q})$.*
4. *$KS_{cl}(KS_{cl}(\mathfrak{P})) = KS_{cl}(\mathfrak{P})$ and $KS_{int}(KS_{int}(\mathfrak{P})) = KS_{int}(\mathfrak{P})$.*
5. *$KS_{cl}(\mathfrak{P} \cup \mathfrak{Q}) \supseteq KS_{cl}(\mathfrak{P}) \cup KS_{cl}(\mathfrak{Q})$.*
6. *$KS_{int}(\mathfrak{P} \cup \mathfrak{Q}) \supseteq KS_{int}(\mathfrak{P}) \cup KS_{int}(\mathfrak{Q})$.*
7. *$KS_{cl}(\mathfrak{P} \cap \mathfrak{Q}) \subseteq KS_{cl}(\mathfrak{P}) \cap KS_{cl}(\mathfrak{Q})$.*
8. *$KS_{int}(\mathfrak{P} \cap \mathfrak{Q}) \subseteq KS_{int}(\mathfrak{P}) \cap KS_{int}(\mathfrak{Q})$.*
9. *$KS_{cl}(\mathfrak{P}) = [KS_{int}(\mathfrak{P})]^c$.*
10. *$KS_{int}(\mathfrak{P}) = [KS_{cl}(\mathfrak{P})]^c$.*

Example 3.6. *Let $\mathfrak{A} = \{\Upsilon, \Omega, \Psi, \Phi, \Gamma\}$ with $\mathfrak{A}/\mathfrak{R} = \{\{\Upsilon\}, \{\Omega, \Psi, \Phi\}, \{\Gamma\}\}$ and $\mathfrak{X} = \{\Upsilon, \Gamma\} \subseteq \mathfrak{A}$. Then $\mathfrak{I}_{\mathfrak{R}}(\mathfrak{X}) = \{\emptyset, \mathfrak{A}, \{\Upsilon, \Gamma\}\}$. If we consider $S = \{\Omega, \Gamma\}$ and $S' = \{\Upsilon, \Psi, \Phi\}$, then $KS_{\mathfrak{R}}(\mathfrak{X}) = \{\emptyset, \{\Upsilon\}, \{\Gamma\}, \{\Upsilon, \Gamma\}, \{\Omega, \Gamma\}, \{\Upsilon, \Psi, \Phi\}, \{\Upsilon, \Omega, \Gamma\}, \{\Upsilon, \Psi, \Phi, \Gamma\}, \mathfrak{A}\}$.*

4. Kasaj-pre-open sets

Definition 4.1. *Let $(\mathfrak{A}, \mathfrak{I}_{\mathfrak{R}}(\mathfrak{X}), KS_{\mathfrak{R}}(\mathfrak{X}))$ be a Kasaj topological space and $\mathfrak{P} \subseteq \mathfrak{A}$. Then $\mathfrak{P}$ is called Kasaj-pre-open(KS-pre-open) set if $\mathfrak{P} \subseteq KS_{int}(KS_{cl}(\mathfrak{P}))$ and Kasaj-pre-closed(KS-pre-closed) set if $KS_{cl}(KS_{int}(\mathfrak{P})) \subseteq \mathfrak{P}$. The set of all Kasaj-pre-open and Kasaj-pre-closed sets are denoted by $KSPO(\mathfrak{A}, \mathfrak{X})$ and $KSPCL(\mathfrak{A}, \mathfrak{X})$, respectively.*

Theorem 4.2. *$KS_{\mathfrak{R}}(\mathfrak{X}) \subseteq KSPO(\mathfrak{A}, \mathfrak{X})$.*

Proof. Let $\mathfrak{P} \in KS_{\mathfrak{R}}(\mathfrak{X})$, i.e., $\mathfrak{P} = KS_{int}(\mathfrak{P})$. Since $\mathfrak{P} \subseteq KS_{cl}(\mathfrak{P})$ for all subset $\mathfrak{P}$ of $\mathfrak{A}$, therefore, $\mathfrak{P} = KS_{int}(\mathfrak{P}) \subseteq KS_{int}(KS_{cl}(\mathfrak{P}))$, which implies that $\mathfrak{P} \subseteq KS_{int}(KS_{cl}(\mathfrak{P}))$. Therefore $\mathfrak{P} \in KSPO(\mathfrak{A}, \mathfrak{X})$. □

Remark 4.3. *In general, $KSPO(\mathfrak{A}, \mathfrak{X}) \not\subseteq KS_{\mathfrak{R}}(\mathfrak{X})$ (See Example 4.4).*

Example 4.4. *Let $\mathfrak{A} = \{\Upsilon, \Omega, \Psi, \Phi, \Gamma\}$ with $\mathfrak{A}/\mathfrak{R} = \{\{\Upsilon\}, \{\Omega, \Psi, \Phi\}, \{\Gamma\}\}$ and $\mathfrak{X} = \{\Phi, \Gamma\} \subseteq \mathfrak{A}$. Then $\mathfrak{I}_{\mathfrak{R}}(\mathfrak{X}) = \{\emptyset, \mathfrak{A}, \{\Gamma\}, \{\Omega, \Psi, \Phi\}, \{\Omega, \Psi, \Phi, \Gamma\}\}$. If we consider $S = \{\Upsilon, \Omega, \Gamma\}$ and $S' = \{\Psi, \Phi\}$, then*

- *$KS_{\mathfrak{R}}(\mathfrak{X}) = \{\emptyset, \{\Omega\}, \{\Gamma\}, \{\Psi, \Phi\}, \{\Omega, \Gamma\}, \{\Upsilon, \Omega, \Gamma\}, \{\Psi, \Phi, \Gamma\}, \{\Omega, \Psi, \Phi\}, \{\Omega, \Psi, \Phi, \Gamma\}, \mathfrak{A}\}$.*

- $KSPO(\mathfrak{A},\mathfrak{X}) = \{\emptyset, \{\Omega\}, \{\Gamma\}, \{\Psi\}, \{\Phi\}, \{\Psi,\Phi\}, \{\Omega,\Gamma\}, \{\Omega,\Psi\}, \{\Omega,\Phi\}, \{\Phi,\Gamma\}, \{\Omega,\Psi,\Gamma\}, \{\Omega,\Phi,\Gamma\}, \{\Psi,\Gamma\}, \{\Upsilon,\Omega,\Gamma\}, \{\Psi,\Phi,\Gamma\}, \{\Omega,\Psi,\Phi\}, \{\Upsilon,\Omega,\Psi,\Gamma\}, \{\Upsilon,\Omega,\Phi,\Gamma\}, \{\Omega,\Psi,\Phi,\Gamma\}, \mathfrak{A}\}$

One can easily see that $\{\Omega,\Psi\} \in KSPO(\mathfrak{A},\mathfrak{X})$ *but not in* $KS_{\mathfrak{R}}(\mathfrak{X})$.

Theorem 4.5. $KSCL(\mathfrak{X}) \subseteq KSPCL(\mathfrak{A},\mathfrak{X})$.

Proof. Let $\mathfrak{P} \in KSCL(\mathfrak{X})$.(i.e., $KS_{cl}(\mathfrak{P}) = \mathfrak{P}$). Then we have $KS_{cl}(KS_{cl}(\mathfrak{P})) \subseteq \mathfrak{P}$. Since

$$KS_{int}(\mathfrak{P}) \subseteq \mathfrak{P}$$

and

$$KS_{cl}(KS_{int}(\mathfrak{P})) \subseteq KS_{cl}(KS_{cl}(\mathfrak{P})).$$

So, it follows that

$$(KS_{cl}(KS_{int}(\mathfrak{P})) \subseteq KS_{cl}(KS_{cl}(\mathfrak{P})) \subseteq \mathfrak{P}.$$

Hence $\mathfrak{P} \in KSPCL(\mathfrak{A},\mathfrak{X})$. □

Remark 4.6. *In general,* $KSPCL(\mathfrak{A},\mathfrak{X}) \nsubseteq KSCL(\mathfrak{X})$. *Consider Example 4.4, One can see that* $\{\Upsilon,\Phi,\Gamma\}$ *is in* $KSPCL(\mathfrak{A},\mathfrak{X})$ *but not in* $KSCL(\mathfrak{X})$.

Definition 4.7. *The Kasaj-pre-closure and the Kasaj-pre-interior of a set* $\mathfrak{P}$ *is denoted by* $KS\text{-}pre_{cl}(\mathfrak{P})$ *and* $KS-pre_{int}(\mathfrak{P})$*, respectively. It is defined by*

$$KS-pre_{cl}(\mathfrak{P}) = \cap\{\mathfrak{Q} : \mathfrak{P} \subseteq \mathfrak{Q}, \mathfrak{Q} \text{ is KS-pre-closed}\}$$

$$KS-pre_{int}(\mathfrak{P}) = \cup\{\mathfrak{Q} : \mathfrak{Q} \subseteq \mathfrak{P}, \mathfrak{Q} \text{ is KS-pre-open}\}.$$

Remark 4.8.

1. *KS-pre*$_{int}(\mathfrak{P})$ *is the largest KS-pre-open set contained in* $\mathfrak{P}$.
2. *KS-pre*$_{cl}(\mathfrak{P})$ *is the smallest KS-pre-closed set containing* $\mathfrak{P}$.

Theorem 4.9.

1. $\cup_{\alpha\in\Lambda}\mathfrak{P}_\alpha \in KSPO(\mathfrak{A},\mathfrak{X})$ *whenever* $\mathfrak{P}_\alpha \in KSPO(\mathfrak{A},\mathfrak{X})$ *and* Λ *is an index set.*
2. $\cap_{\alpha\in\Lambda}\mathfrak{P}_\alpha \in KSPCL(\mathfrak{A},\mathfrak{X})$ *whenever* $\mathfrak{P}_\alpha \in KSPCL(\mathfrak{A},\mathfrak{X})$ *and* Λ *is an index set.*

Proof. (1.) Let $\{\mathfrak{P}_\alpha : \alpha \in I\} \subseteq KSPO(\mathfrak{A},\mathfrak{X})$. By definition of KS-pre-open set, for each α, $\mathfrak{P}_\alpha \subseteq KS_{int}(KS_{cl}(\mathfrak{P}_\alpha))$, which implies that

$$\begin{aligned}\cup_\alpha\mathfrak{P}_\alpha &\subseteq \cup_\alpha KS_{int}(KS_{cl}(\mathfrak{P}_\alpha))\\ &\subseteq KS_{int}(\cup_\alpha KS_{cl}(\mathfrak{P}_\alpha))\\ &\subseteq KS_{int}(KS_{cl}(\cup_\alpha(\mathfrak{P}_\alpha)))\end{aligned}$$

Hence $\cup_\alpha\mathfrak{P}_\alpha \in KSPO(\mathfrak{A},\mathfrak{X})$.

(2.) Let $\{\mathfrak{P}_\alpha : \alpha \in I\} \subseteq KSPCL(\mathfrak{A},\mathfrak{X})$. By definition of KS-pre-closed set, for each α,
$KS_{cl}(KS_{int}(\mathfrak{P}_\alpha)) \subseteq \mathfrak{P}_\alpha$.
Now

$$\begin{aligned}KS_{cl}(KS_{int}(\cap_\alpha(\mathfrak{P}_\alpha))) &\subseteq KS_{cl}(\cap_\alpha(KS_{int}(\mathfrak{P}_\alpha))\\ &\subseteq \cap_\alpha(KS_{cl}(KS_{int}(\mathfrak{P}_\alpha)))\\ &\subseteq \cap_\alpha\mathfrak{P}_\alpha\end{aligned}$$

Hence, $\cap_\alpha\mathfrak{P}_\alpha \in KSPCL(\mathfrak{A},\mathfrak{X})$. □

Theorem 4.10.

1. $\mathfrak{P} = KS\text{-}pre_{cl}(\mathfrak{P})$ *iff* $\mathfrak{P} \in KSPCL(\mathfrak{A},\mathfrak{X})$.
2. $\mathfrak{P} = KS\text{-}pre_{int}(\mathfrak{P})$ *iff* $\mathfrak{P} \in KSPO(\mathfrak{A},\mathfrak{X})$.

Proof.

1. ($\Rightarrow$) Assume that $\mathfrak{P} \in KSPCL(\mathfrak{A},\mathfrak{X})$, $\mathfrak{P} \subseteq KS-pre_{cl}(\mathfrak{P})$ and $KS-pre_{cl}(\mathfrak{P}) = \cap\{\mathfrak{Q} : \mathfrak{P} \subseteq \mathfrak{Q}, \mathfrak{Q}$ is $KS-pre-$ closed set$\}$. Since $\mathfrak{P} \in KSPCL(\mathfrak{A},\mathfrak{X})$. $\mathfrak{P}$ is an element of $\cap\{\mathfrak{Q} : \mathfrak{P} \subseteq \mathfrak{Q}, \mathfrak{Q}$ is $KS-pre-$ closed set$\}$. So, $\cap\{\mathfrak{Q} : \mathfrak{P} \subseteq \mathfrak{Q}, \mathfrak{Q}$ is $KS-pre-$ closed set$\} = \mathfrak{P}$. Hence $\mathfrak{P} = KS-pre_{cl}(\mathfrak{P})$.

   ($\Leftarrow$) Assume that $\mathfrak{P} = KS$-pre$_{cl}(\mathfrak{P})$. Then by Remark 4.8, $\mathfrak{P} \in KSPCL(\mathfrak{A},\mathfrak{X})$.

2. ($\Rightarrow$) Assume that $\mathfrak{P} \in KSPO(\mathfrak{A},\mathfrak{X})$. Then $\mathfrak{P} \subseteq \cup\{\mathfrak{Q} : \mathfrak{Q} \subseteq \mathfrak{P}, \mathfrak{Q}$ is $KS-pre-$ open set$\} = KS-pre_{int}(\mathfrak{P})$. So, $\mathfrak{P} \subseteq KS-pre_{int}(\mathfrak{P})$. As $\mathfrak{P} \in KSPO(\mathfrak{A},\mathfrak{X})$, $KS-pre_{int}(\mathfrak{P}) \subseteq \mathfrak{P}$. Hence $\mathfrak{P} = KS-pre_{int}(\mathfrak{P})$.

   ($\Leftarrow$) Assume that $\mathfrak{P} = KS-pre_{int}(\mathfrak{P})$. Then by Remark 4.8, $\mathfrak{P} \in KSPO(\mathfrak{A},\mathfrak{X})$.
   Hence, we get desired.

□

5. Kasaj-semi-open sets

Definition 5.1. *Let* $(\mathfrak{A}, \mathfrak{I}_{\mathfrak{R}}(\mathfrak{X}), KS_{\mathfrak{R}}(\mathfrak{X}))$ *be a Kasaj topological space and* $\mathfrak{P} \subseteq \mathfrak{A}$. *Then* $\mathfrak{P}$ *is called Kasaj-semi-open(KS-semi-open) set if* $\mathfrak{P} \subseteq KS_{cl}(KS_{int}(\mathfrak{P}))$ *and Kasaj-semi-closed(KS-semi-closed) set if* $KS_{int}(KS_{cl}(\mathfrak{P})) \subseteq \mathfrak{P}$. *The set of all Kasaj-semi-open sets is denoted by* $KSSO(\mathfrak{A},\mathfrak{X})$ *and similarly, The set of all Kasaj-semi-closed sets is denoted by* $KSSCL(\mathfrak{A},\mathfrak{X})$.

Theorem 5.2. $KS_{\mathfrak{R}}(\mathfrak{X}) \subseteq KSSO(\mathfrak{A},\mathfrak{X})$.

Proof. Let $\mathfrak{P} \in KS_{\mathfrak{R}}(\mathfrak{X})$, (i.e., $\mathfrak{P} = KS_{int}(\mathfrak{P})$). Since $\mathfrak{P} \subseteq KS_{cl}(\mathfrak{P})$ for all subset $\mathfrak{P}$ of $\mathfrak{A}$, therefore, $\mathfrak{P} = KS_{int}(\mathfrak{P}) \subseteq KS_{cl}(KS_{int}(\mathfrak{P}))$, which implies that $\mathfrak{P} \subseteq KS_{cl}(KS_{int}(\mathfrak{P}))$. Therefore $\mathfrak{P} \in KSSO(\mathfrak{A},\mathfrak{X})$. □

Remark 5.3. *In general* $KSSO(\mathfrak{A},\mathfrak{X}) \nsubseteq KS_{\mathfrak{R}}(\mathfrak{X})$ *(See Example 5.4).*

Example 5.4. *Let* $\mathfrak{A} = \{\Upsilon,\Omega,\Psi,\Phi,\Gamma\}$ *with* $\mathfrak{A}/\mathfrak{R} = \{\{\Upsilon\}, \{\Omega,\Psi\}, \{\Phi,\Gamma\}\}$ *and* $\mathfrak{X} = \{\Upsilon,\Psi\} \subseteq \mathfrak{A}$. *Then* $\mathfrak{I}_{\mathfrak{R}}(\mathfrak{X}) = \{\emptyset, \mathfrak{A}, \{\Upsilon\}, \{\Omega,\Psi\}, \{\Upsilon,\Omega,\Psi\}\}$. *If we consider* $S = \{\Upsilon,\Omega,\Phi\}$ *and* $S' = \{\Psi,\Gamma\}$*, then*

- $KS_{\mathfrak{R}}(\mathfrak{X}) = \{\emptyset, \{\Upsilon\}, \{\Omega\}, \{\Psi\}, \{\Psi,\Gamma\}, \{\Upsilon,\Psi\}, \{\Upsilon,\Omega\}, \{\Omega,\Psi\}, \{\Upsilon,\Omega,\Phi\}, \{\Upsilon,\Omega,\Psi\}, \{\Omega,\Psi,\Gamma\}, \{\Upsilon,\Psi,\Gamma\}, \{\Upsilon,\Omega,\Psi,\Phi\}, \{\Upsilon,\Omega,\Psi,\Gamma\}, \mathfrak{A}\}$.

- $KSSO(\mathfrak{A},\mathfrak{X}) = \{\emptyset, \{\Upsilon\}, \{\Omega\}, \{\Psi\}, \{\Psi,\Gamma\}, \{\Upsilon,\Psi\}, \{\Upsilon,\Omega\}, \{\Upsilon,\Phi\}, \{\Omega,\Phi\}, \{\Omega,\Psi\}, \{\Upsilon,\Omega,\Phi\}, \{\Upsilon,\Omega,\Psi\}, \{\Upsilon,\Psi,\Phi\}, \{\Omega,\Psi,\Phi\}, \{\Omega,\Psi,\Gamma\}, \{\Upsilon,\Psi,\Gamma\}, \{\Upsilon,\Omega,\Psi,\Phi\}, \{\Upsilon,\Omega,\Psi,\Gamma\}, \{\Upsilon,\Psi,\Phi,\Gamma\}, \{\Omega,\Psi,\Phi,\Gamma\}, \mathfrak{A}\}$.

One can easily see that $\{\Upsilon,\Phi\}$ is in $KSSO(\mathfrak{A},\mathfrak{X})$ but not in $KS_{\mathfrak{R}}(\mathfrak{X})$.

Theorem 5.5. *$KSCL(\mathfrak{X}) \subseteq KSSCL(\mathfrak{A},\mathfrak{X})$.*

Proof. Let $\mathfrak{P} \in KSCL(\mathfrak{X})$.(i.e., $KS_{cl}(\mathfrak{P}) = \mathfrak{P}$). Then we have $KS_{cl}(KS_{cl}(\mathfrak{P})) \subseteq \mathfrak{P}$. Since

$$KS_{int}(\mathfrak{P}) \subseteq \mathfrak{P}$$

and

$$KS_{int}(KS_{cl}(\mathfrak{P})) \subseteq KS_{cl}(KS_{cl}(\mathfrak{P})).$$

So, it follows that

$$(KS_{int}(KS_{cl}(\mathfrak{P})) \subseteq KS_{cl}(KS_{cl}(\mathfrak{P})) \subseteq \mathfrak{P}.$$

Hence $\mathfrak{P} \in KSSCL(\mathfrak{A},\mathfrak{X})$. □

Remark 5.6. *In general, $KSSCL(\mathfrak{A},\mathfrak{X}) \nsubseteq KSCL(\mathfrak{X})$.Consider Example 5.4, One can see that $\{\Omega\}$ is in $KSSCL(\mathfrak{A},\mathfrak{X})$ but not in $KSCL(\mathfrak{X})$.*

Definition 5.7. *The Kasaj-semi-closure and the Kasaj-semi-interior of a set $\mathfrak{P}$ are denoted by KS-semi$_{cl}(\mathfrak{P})$ and KS-semi$_{int}(\mathfrak{P})$, respectively. They are defined by*

$$KS\text{-}semi_{cl}(\mathfrak{P}) = \cap\{\mathfrak{Q} : \mathfrak{P} \subseteq \mathfrak{Q}, \mathfrak{Q} \text{ is KS-semi-closed}\}$$

$$KS\text{-}semi_{int}(\mathfrak{P}) = \cup\{\mathfrak{Q} : \mathfrak{Q} \subseteq \mathfrak{P}, \mathfrak{Q} \text{ is KS-semi-open}\}.$$

Remark 5.8.

1. *KS-semi$_{int}(\mathfrak{P})$ is the largest KS-semi-open set contained in $\mathfrak{P}$.*
2. *KS-semi$_{cl}(\mathfrak{P})$ is the smallest KS-semi-closed set containing $\mathfrak{P}$.*

Theorem 5.9.

1. *$\cup_{\alpha\in\Lambda}\mathfrak{P}_\alpha \in KSSO(\mathfrak{A},\mathfrak{X})$ whenever $\mathfrak{P}_\alpha \in KSSO(\mathfrak{A},\mathfrak{X})$ and Λ is an index set.*
2. *$\cap_{\alpha\in\Lambda}\mathfrak{P}_\alpha \in KSSCL(\mathfrak{A},\mathfrak{X})$ whenever $\mathfrak{P}_\alpha \in KSSCL(\mathfrak{A},\mathfrak{X})$ and Λ is an index set.*

Proof. (1.) Let $\{\mathfrak{P}_\alpha : \alpha \in I\} \subseteq KSSO(\mathfrak{A},\mathfrak{X})$. By definition of KS-semi-open set, for each α, $\mathfrak{P}_\alpha \subseteq KS_{cl}(KS_{int}(\mathfrak{P}_\alpha))$, which implies that

$$\begin{aligned}\cup_\alpha\mathfrak{P}_\alpha &\subseteq \cup_\alpha KS_{cl}(KS_{int}(\mathfrak{P}_\alpha))\\ &\subseteq KS_{cl}(\cup_\alpha KS_{int}(\mathfrak{P}_\alpha))\\ &\subseteq KS_{cl}(KS_{int}(\cup_\alpha(\mathfrak{P}_\alpha)))\end{aligned}$$

Hence $\cup_\alpha\mathfrak{P}_\alpha \in KSSO(\mathfrak{A},\mathfrak{X})$.
(2.) Let $\{\mathfrak{P}_\alpha : \alpha \in I\} \subseteq KSSCL(\mathfrak{A},\mathfrak{X})$. By definition of KS-semi-closed set, for each α,

$$KS_{int}(KS_{cl}(\mathfrak{P}_\alpha)) \subseteq \mathfrak{P}_\alpha.$$

$$\begin{aligned}\textit{Now}\quad KS_{int}(KS_{cl}(\cap_\alpha(\mathfrak{P}_\alpha))) &\subseteq KS_{int}(\cap_\alpha(KS_{cl}(\mathfrak{P}_\alpha))\\ &\subseteq \cap_\alpha(KS_{int}(KS_{cl}(\mathfrak{P}_\alpha)))\\ &\subseteq \cap_\alpha\mathfrak{P}_\alpha\end{aligned}$$

Hence, $\cap_\alpha\mathfrak{P}_\alpha \in KSSCL(\mathfrak{A},\mathfrak{X})$. □

Remark 5.10.

1. *If $\mathfrak{P},\mathfrak{Q} \in KSSO(\mathfrak{A},\mathfrak{X})$ but in general $\mathfrak{P}\cap\mathfrak{Q}$ need not in $KSSO(\mathfrak{A},\mathfrak{X})$. In example 5.4, $\{\Upsilon,\Phi\}$, $\{\Omega,\Phi\} \in KSSO(\mathfrak{A},\mathfrak{X})$ but $\{\Upsilon,\Phi\}\cap\{\Omega,\Phi\} = \{\Phi\} \notin KSSO(\mathfrak{A},\mathfrak{X})$.*
2. *If $\mathfrak{P},\mathfrak{Q} \in KSSCL(\mathfrak{A},\mathfrak{X})$ but in general $\mathfrak{P}\cup\mathfrak{Q}$ need not in $KSSCL(\mathfrak{A},\mathfrak{X})$. In example 5.4, $\{\Omega,\Psi,\Gamma\}, \{\Upsilon,\Psi,\Gamma\} \in KSSCL(\mathfrak{A},\mathfrak{X})$ but $\{\Omega,\Psi,\Gamma\}\cup\{\Upsilon,\Psi,\Gamma\} = \{\Upsilon,\Omega,\Psi,\Gamma\} \notin KSSCL(\mathfrak{U},\mathfrak{X})$.*

Theorem 5.11.

1. *$\mathfrak{P} = KS\text{-}semi_{cl}(\mathfrak{P})$ iff $\mathfrak{P} \in KSSCL(\mathfrak{U},\mathfrak{X})$.*
2. *$\mathfrak{P} = KS\text{-}semi_{int}(\mathfrak{P})$ iff $\mathfrak{P} \in KSSO(\mathfrak{U},\mathfrak{X})$.*

Proof.

1. ($\Rightarrow$) Assume that $\mathfrak{P} \in KSSCL(\mathfrak{U},\mathfrak{X})$, $\mathfrak{P} \subseteq KS-semi_{cl}(\mathfrak{P})$ and $KS-semi_{cl}(\mathfrak{P}) = \cap\{\mathfrak{Q} : \mathfrak{P} \subseteq \mathfrak{Q},\ \ \mathfrak{Q}$ is $KS-semi-$ closed set$\}$. Since $\mathfrak{P} \in KSSCL(\mathfrak{U},\mathfrak{X})$. $\mathfrak{P}$ is an element of $\{\mathfrak{Q} : \mathfrak{P} \subseteq \mathfrak{Q}, \mathfrak{Q}$ is $KS-semi-$ closed set$\}$. So,

   $$\cap\{\mathfrak{Q} : \mathfrak{P} \subseteq \mathfrak{Q}, \mathfrak{Q} \text{ is } KS-semi- \text{ closed set}\} = \mathfrak{P}.$$

   Hence $\mathfrak{P} = KS-semi_{cl}(\mathfrak{P})$.

   ($\Leftarrow$) Assume that $\mathfrak{P} = KS-semi_{cl}(\mathfrak{P})$. Then by Remark 5.8, $\mathfrak{P} \in KSSCL(\mathfrak{U},\mathfrak{X})$.

2. ($\Rightarrow$) Assume that $\mathfrak{P} \in KSSO(\mathfrak{U},\mathfrak{X})$. Then

   $$\begin{aligned}\mathfrak{P} &\subseteq \cup\{\mathfrak{Q} : \mathfrak{Q} \subseteq \mathfrak{P}, \mathfrak{Q} \text{ is } KS-semi- \text{ open set}\}\\ &= KS-semi_{int}(\mathfrak{P}).\end{aligned}$$

   So, $\mathfrak{P} \subseteq KS-semi_{int}(\mathfrak{P})$. As $\mathfrak{P} \in KSSO(\mathfrak{U},\mathfrak{X})$, $KS-semi_{int}(\mathfrak{P}) \subseteq \mathfrak{P}$. Hence $\mathfrak{P} = KS-semi_{int}(\mathfrak{P})$.

   ($\Leftarrow$) Assume that $\mathfrak{P} = KS-semi_{int}(\mathfrak{P})$. Then by Remark 5.8, $\mathfrak{P} \in KSSO(\mathfrak{U},\mathfrak{X})$. Hence, we get desired. □

Example 5.12. *Let $\mathfrak{A} = \{\Upsilon,\Omega,\Psi,\Phi,\Gamma\}$ with $\mathfrak{A}/\mathfrak{R} = \{\{\Upsilon\}, \{\Omega,\Psi,\Phi\}, \{\Gamma\}\}$ and $\mathfrak{X} = \{\Phi,\Gamma\} \subseteq \mathfrak{A}$. Then $\mathfrak{I}_{\mathfrak{R}}(\mathfrak{X}) = \{\emptyset, \mathfrak{A}, \{\Gamma\}, \{\Omega,\Psi,\Phi\}, \{\Omega,\Psi,\Phi,\Gamma\}\}$. If we consider $S = \{\Upsilon,\Omega,\Gamma\}$ and $S' = \{\Psi,\Phi\}$, then*

- $KS_{\mathfrak{R}}(\mathfrak{X}) = \{\emptyset, \{\Omega\}, \{\Gamma\}, \{\Psi,\Phi\}, \{\Omega,\Gamma\}, \{\Upsilon,\Omega,\Gamma\}, \{\Psi,\Phi,\Gamma\}, \{\Omega,\Psi,\Phi\}, \{\Omega,\Psi,\Phi,\Gamma\}, \mathfrak{A}\}$.

- $KSSO(\mathfrak{A},\mathfrak{X}) = \{\emptyset, \{\Omega\}, \{\Gamma\}, \{\Upsilon,\Omega\}, \{\Upsilon,\Gamma\}, \{\Psi,\Phi\}, \{\Omega,\Gamma\}, \{\Upsilon,\Omega,\Gamma\}, \{\Psi,\Phi,\Gamma\}, \{\Omega,\Psi,\Phi\}, \{\Omega,\Psi,\Phi,\Gamma\}, \{\Upsilon,\Omega,\Psi,\Phi\}, \{\Upsilon,\Psi,\Phi,\Gamma\}, \mathfrak{A}\}$.
- $KSPO(\mathfrak{A},\mathfrak{X}) = \{\emptyset, \{\Omega\}, \{\Gamma\}, \{\Psi\}, \{\Phi\}, \{\Psi,\Phi\}, \{\Omega,\Gamma\}, \{\Omega,\Psi\}, \{\Omega,\Phi\}, \{\Phi,\Gamma\}, \{\Omega,\Psi,\Gamma\}, \{\Omega,\Phi,\Gamma\}, \{\Upsilon,\Omega,\Gamma\}, \{\Upsilon,\Omega,\Psi,\Gamma\}, \{\Upsilon,\Omega,\Phi,\Gamma\}, \{\Psi,\Phi,\Gamma\}, \{\Omega,\Psi,\Phi\}, \{\Omega,\Psi,\Phi,\Gamma\}, \mathfrak{A}\}$.

Remark 5.13.

- $KSPO(\mathfrak{U},\mathfrak{X}) \nsubseteq KSSO(\mathfrak{U},\mathfrak{X})$. *In example 5.12* $\{\Phi\}$ *is in* $KSPO(\mathfrak{U},\mathfrak{X})$ *but not in* $KSSO(\mathfrak{U},\mathfrak{X})$.
- $KSPO(\mathfrak{U},\mathfrak{X}) \nsupseteq KSSO(\mathfrak{U},\mathfrak{X})$. *In example 5.12* $\{\Upsilon,\Omega\}$ *is in* $KSSO(\mathfrak{A},\mathfrak{X})$ *but not in* $KSPO(\mathfrak{A},\mathfrak{X})$.

6. Kasaj-continuous functions

We first define Kasaj-continuous (KS-continuous) functions.

Definition 6.1. *Let* $(\mathfrak{A}, \mathfrak{I}_{\mathfrak{R}}(\mathfrak{X}), KS_{\mathfrak{R}}(\mathfrak{X}))$ *and* $(\mathfrak{V}, \mathfrak{I}_{\mathfrak{R}'}(\mathfrak{Y}), KS_{\mathfrak{R}'}(\mathfrak{Y}))$ *be two Kasaj topological spaces and* $\mathfrak{X} \subseteq \mathfrak{A}$ *and* $\mathfrak{Y} \subseteq \mathfrak{V}$. *Then* $f : \mathfrak{A} \to \mathfrak{V}$ *is Kasaj-continuous (KS-continuous) function if* $f^{-1}(D) \in KS_{\mathfrak{R}}(\mathfrak{X})$ *whenever* $D \in KS_{\mathfrak{R}}(\mathfrak{Y})$.

Theorem 6.2. *Let* $(\mathfrak{A}, \mathfrak{I}_{\mathfrak{R}}(\mathfrak{X}), KS_{\mathfrak{R}}(\mathfrak{X}))$ *and* $(\mathfrak{V}, \mathfrak{I}_{\mathfrak{R}'}(\mathfrak{Y}), KS_{\mathfrak{R}'}(\mathfrak{Y}))$ *be two Kasaj topological spaces and* $\mathfrak{X} \subseteq \mathfrak{A}$ *and* $\mathfrak{Y} \subseteq \mathfrak{V}$. *Then* $f : \mathfrak{A} \to \mathfrak{V}$ *is KS-continuous function if and only if* $f^{-1}(D) \in KSCL(\mathfrak{X})$ *whenever* $D \in KSCL(\mathfrak{Y})$.

Proof. Let, $f : \mathfrak{A} \to \mathfrak{V}$ is KS-continuous function and $D \in KSCL(\mathfrak{Y})$. Then $D^c \in KS_{\mathfrak{R}}(\mathfrak{Y})$. By hypothesis $f^{-1}(D^c) \in KS_{\mathfrak{R}}(\mathfrak{X})$, i.e. $[f^{-1}(D)]^c \in KS_{\mathfrak{R}}(\mathfrak{X})$. Hence $f^{-1}(D) \in KSCL(\mathfrak{X})$ whenever $D \in KSCL(\mathfrak{Y})$.
Conversely suppose $[f^{-1}(D)]^c \in KS_{\mathfrak{R}}(\mathfrak{X})$ whenever $D \in KSCL(\mathfrak{Y})$. Let $D \in KS_{\mathfrak{R}}(\mathfrak{Y})$ then $D^c \in KSCL(\mathfrak{Y})$. By assumption $f^{-1}(D^c) \in KSCL(\mathfrak{X})$. i.e. $[f^{-1}(D)]^c \in KSCL(\mathfrak{X})$. Then $f^{-1}(D) \in KS_{\mathfrak{R}}(\mathfrak{X})$. Hence f is KS-continuous. □

Definition 6.3. *Let* $(\mathfrak{A}, \mathfrak{I}_{\mathfrak{R}}(\mathfrak{X}), KS_{\mathfrak{R}}(\mathfrak{X}))$ *and* $(\mathfrak{V}, \mathfrak{I}_{\mathfrak{R}'}(\mathfrak{Y}), KS_{\mathfrak{R}'}(\mathfrak{Y}))$ *be two Kasaj topological spaces and* $\mathfrak{X} \subseteq \mathfrak{A}$ *and* $\mathfrak{Y} \subseteq \mathfrak{V}$. *Then* $f : \mathfrak{A} \to \mathfrak{V}$ *is KS-pre-continuous function if* $f^{-1}(D) \in KSPCL(\mathfrak{A},\mathfrak{X})$ *whenever* $D \in KSCL(\mathfrak{Y})$.

Theorem 6.4. *Every KS-continuous function is KS-pre- continuous function.*

Proof. Let $f : \mathfrak{A} \to \mathfrak{V}$ be a KS-continuous function, *i.e.* $f^{-1}(D) \in KSCL(\mathfrak{X})$ whenever $D \in KSCL(\mathfrak{Y})$. By Theorem 4.2, Therefore $f^{-1}(D) \in KSPCL(\mathfrak{A},\mathfrak{X})$ for all $D \in KSCL(\mathfrak{Y})$. Hence, f is KS-pre-continuous function. □

Definition 6.5. *Let* $(\mathfrak{A}, \mathfrak{I}_{\mathfrak{R}}(\mathfrak{X}), KS_{\mathfrak{R}}(\mathfrak{X}))$ *and* $(\mathfrak{V}, \mathfrak{I}_{\mathfrak{R}'}(\mathfrak{Y}), KS_{\mathfrak{R}'}(\mathfrak{Y}))$ *be two Kasaj topological spaces and* $\mathfrak{X} \subseteq \mathfrak{A}$ *and* $\mathfrak{Y} \subseteq \mathfrak{V}$. *Then* $f : \mathfrak{A} \to \mathfrak{V}$ *is KS-semi-continuous function if* $f^{-1}(D) \in KSSCL(\mathfrak{A},\mathfrak{X})$ *whenever* $D \in KSCL(\mathfrak{Y})$.

Theorem 6.6. *Every KS-continuous function is KS- semi-continuous function.*

Proof. Let $f : \mathfrak{A} \to \mathfrak{V}$ be a KS-continuous function, *i.e.* $f^{-1}(D) \in KSCL(\mathfrak{X})$ whenever $D \in KSCL(Y)$. By Theorem 5.2, Therefore $f^{-1}(D) \in KSSCL(\mathfrak{A},\mathfrak{X})$ for all $D \in KSCL(\mathfrak{Y})$. Hence, f is KS-semi-continuous function. □

Conclusion

In this paper, some of the properties of these new classes are discussed and we get the following inversion :

$$KSPO(\mathfrak{A},\mathfrak{X}) \supsetneq KS_{\mathfrak{R}}(\mathfrak{X}) \subsetneq KSSO(\mathfrak{A},\mathfrak{X})$$

we have shown that none of implication is reversible. This shall be extended in future research with some applications.

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