International Journal of Analysis and Applications

Best Proximity Point for G-Generalized $\zeta - \beta - T$ Contraction

Amit Duhan[1], Manoj Kumar[1], Savita Rathee[2], Monika Swami[2,*]

[1] *Baba Masthnath University, Rohtak, 124001, India*
[2] *Maharshi Dayanand University, Rohtak, 124001, India*

[*] *Corresponding author:* monikaswami06@gmail.com

Abstract. In this paper, we find the best proximity point in G-metric spaces for G-generalized $\zeta - \beta - T$ contraction mappings and verify the existence and uniqueness of the best proximity point in the complete G metric space using the idea of an approximatively compact set. In addition, an example is provided to illustrate the outcome.

1. Introduction

The "fixed point theory" developed as an important tool for finding the solution of equation of the type $Tx = x$, where T is a self-mapping defined over a subset of a metric space. A difficulty emerges when the mapping shifts to a non-self mapping. The solution to this question is the "best proximity point theory." In this case, a point is calculated having the minimum distance between the point and its image. This point is called “best proximity point" and it reduces to“ fixed point" when the mapping reduces from non-self to self-mapping.

In 2006, Mustafa and Sims [9] popularised a metric space in its generalized form, named as G-metric space. G- metric space is a generalization in which each triplet of elements is allocated a non-negative real number. Physically, this is a measure of mutual distance between three elements taken together. Researchers worked on G-metric space to calculate the fixed point for different type of contractions. As G-metric space becomes a vast area for fixed point theory, but on the other hand, in 2014, Hussain et al. [6] were the first who work on G-metric space to calculate the “best proximity point" for the introduced proximal contraction. Later on, Abbas [5], Chodhury [3], Ansari [1] and researchers work in this direction for calculating “best proximity points" in G-metric spaces [2, 7].

Received: Oct. 17, 2022.
2020 *Mathematics Subject Classification.* 47H10, 54H25, 46J10, 46J15.
Key words and phrases. best proximity point; G-metric space; Geraghty contraction.

https://doi.org/10.28924/2291-8639-21-2023-0

ISSN: 2291-8639

2. Preliminaries

Definition 2.1. *[9] Let X be a nonempty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties:*

(G1) $G(x, y, z) = 0$, if $x = y = z$,
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ xnd $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all variables),
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ (rectangular inequality) for all $x, y, z, a \in X$.

Then the function G is called a generalized metric or G-metric on X and the pair (X, G) is called a G-metric space.

Every G-metric on X generates a metric d_G on X defined by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \forall x, y \in X.$$

Example 2.1. *[9] Let $X = [0, \infty)$. The function $G : X \times X \times X \to [0, \infty)$ defined by $G(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, x \in X$ is a G-metric on X.*

Definition 2.2. *[9] Let (X, G) be G-metric space and let $\{x_m\}$ be a sequence of points of X, then $\{x_m\}$ is G-convergent to $x \in X$ if*

$$\lim_{m,l \to \infty} G(x, x_m, x_l) = 0$$

that is, for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $G(x, x_m, x_l) < \epsilon$, for all $m, l \geq N$. We call x the limit of the sequence and write $x_m \to x$ or $\lim_{m \to \infty} x_m = x$.

Proposition 2.1. *[9] Let (X, G) be a G-metric space. The following statements are equivalent:*

(i) $\{x_m\}$ is G-convergent to x,
(ii) $G(x_m, x_m, x) \to 0$ as $m \to +\infty$,
(iii) $G(x_m, x, x) \to 0$ as $m \to +\infty$,
(iv) $G(x_m, x_l, x) \to 0$ as $m, l \to +\infty$.

Definition 2.3. *[9] Let (X, G) be a G-metric space. A sequence $\{x_m\}$ is called G-Cauchy sequence, if for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_m, x_l, x_k) < \epsilon$ for all $m, l, k \geq N$, that is $G(x_m, x_l, x_k) \to 0$ as $m, l, k \to +\infty$.*

Proposition 2.2. *[9] Let (X, G) be a G-metric space, then following statements are equivalent:*

(i) the sequence $\{x_m\}$ is G-Cauchy,
(ii) for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_m, x_l, x_l) < \epsilon$, for all $m, l \geq N$.
(iii) $\{x_m\}$ is a Cauchy sequence in the metric space (X, d_G).

Definition 2.4. *[9] A G-metric space (X, G) is called a G-complete if every G-Cauchy sequence is G-convergent in (X, G).*

Definition 2.5. *Let (X, G) be a G-metric space. A mapping $F : X \times X \times X \to X$ is said to be continuous if for any three G-convergent sequences $\{x_m\}, \{y_m\}$ and $\{z_m\}$ converging to x, y and z respectively, then $\{F(x_m, y_m, z_m)\}$ is G-convergent to $F(x, y, z)$.*

Lemma 2.1. *[8] From (G5) and (G4), we have*

$$\begin{aligned} \mathrm{G}(x, y, y) = \mathrm{G}(y, y, x) &\leq \mathrm{G}(y, x, x) + \mathrm{G}(x, y, x) \\ &= 2\mathrm{G}(y, x, x). \end{aligned}$$

Definition 2.6. *Let (X, G) be a G-metric space and $\mathcal{Q}$ and $\mathcal{R}$ be two nonempty subsets of a G-metric space (X, G). We define the following sets:*

$$\begin{aligned} \mathcal{Q}_0 =& \{x \in \mathcal{Q} : d_{\mathrm{G}}(x, y) = d_{\mathrm{G}}(\mathcal{Q}, \mathcal{R}) \text{ for some } y \in \mathcal{R}\} \\ \mathcal{R}_0 =& \{y \in \mathcal{R} : d_{\mathrm{G}}(x, y) = d_{\mathrm{G}}(\mathcal{Q}, \mathcal{R}) \text{ for some } x \in \mathcal{Q}\} \end{aligned}$$

where $d_{\mathrm{G}}(\mathcal{Q}, \mathcal{R}) = \inf\{d_{\mathrm{G}}(x, y) : x \in \mathcal{Q}, y \in \mathcal{R}\}$.

Definition 2.7. *[6] Let (X, G) be a G-metric space and let $\mathcal{Q}$ and $\mathcal{R}$ be two nonempty subsets of X. Then $\mathcal{R}$ is said to be approximatively compact with respect to $\mathcal{Q}$ if every sequence $\{y_m\}$ in $\mathcal{R}$, satisfying the condition $d_{\mathrm{G}}(x, y_m) \to d_{\mathrm{G}}(x, \mathcal{R})$ for some x in $\mathcal{Q}$, has a convergent subsequence.*

3. Main Results

Firstly, we contemplate that

$$\begin{aligned} &\Xi = \{\zeta : [0, \infty) \to [0, \infty) \text{ such that } \zeta \text{ is nondecreasing and continuous where } \zeta(x) = 0 \\ &\quad \text{if and only if } x = 0.\} \\ &\Upsilon = \{\beta : [0, \infty) \to [0, 1) \text{ such that } \beta(x_l) \to 1 \text{ then } x_l \to 0\} \end{aligned}$$

Definition 3.1. *Let $\mathcal{Q}$ and $\mathcal{R}$ be two nonempty subsets of a "G-metric space (X, G)", then $T : \mathcal{Q} \to \mathcal{R}$ is said to be G-generalized $\zeta - \beta - T$ contractive mapping if, for $x, u, u^*, y, v \in \mathcal{Q}$ and $L \geq 1$ such that*

$$\begin{aligned} & d_{\mathrm{G}}(u, Tx) = d_{\mathrm{G}}(\mathcal{Q}, \mathcal{R}) \\ & d_{\mathrm{G}}(u^*, Tu) = d_{\mathrm{G}}(\mathcal{Q}, \mathcal{R}) \\ & d_{\mathrm{G}}(v, Ty) = d_{\mathrm{G}}(\mathcal{Q}, \mathcal{R}) \\ \implies & \zeta(\mathrm{G}(u, u^*, v)) \leq \beta(\zeta(M(x, u, y) - d_{\mathrm{G}}(\mathcal{Q}, \mathcal{R}))).\zeta(M(x, u, y) - d_{\mathrm{G}}(\mathcal{Q}, \mathcal{R})) \\ & \qquad + L\zeta[N(x, u, y) - d_{\mathrm{G}}(\mathcal{Q}, \mathcal{R})] \end{aligned} \tag{3.1}$$

where $M(x,u,y) = \max\{G(x,Tx,u), G(x,Tx,y), G(u,Tu,y), G(y,Ty,u), G(x,u,y)\}$ *and* $N(x,u,y) = \min\{G(x,Tx,u), G(u,Tu,y), G(y,Ty,x), G(Tx,u,y)\}$

Theorem 3.1. *Let* $(\mathcal{Q},\mathcal{R})$ *be pair of nonempty closed subset of a "G-metric space* (X,G)*" such that* $(\mathcal{Q},G)$ *is "complete G-metric space" and* $\mathcal{R}$ *is approximatively compact with respect to* $\mathcal{Q}$*. Consider* $T:\mathcal{Q}\to\mathcal{R}$ *be a G-generalized* $\zeta-\beta-T$ *contractive mapping satisfies* $T(\mathcal{Q}_0)\subseteq\mathcal{R}_0$*. Then* T *has a unique "best proximity point" in* $\mathcal{Q}$ *that is,* $q\in\mathcal{Q}$ *such that* $d_G(q,Tq)=d_G(\mathcal{Q},\mathcal{R})$*.*

Proof. Since the subset $\mathcal{Q}_0$ is non-empty subset of $\mathcal{Q}$, we consider $x_0\in\mathcal{Q}_0$ such that $T(x_0)\in T(\mathcal{Q}_0)\subseteq\mathcal{R}_0$, then we can find $x_1\in\mathcal{Q}_0$ such that

$$d_G(x_1,Tx_0)=d_G(\mathcal{Q},\mathcal{R})$$

Thereafter, since $Tx_1\in T\mathcal{Q}_0\subseteq\mathcal{R}_0$, it supervene that there is an element x_2 in $\mathcal{Q}_0$ such that $d_G(x_2,Tx_1)=d_G(\mathcal{Q},\mathcal{R})$. Repeatedly, we get a sequence $\{x_m\}$ in $\mathcal{Q}_0$ satisfying

$$d_G(x_{m+1},Tx_m)=d_G(\mathcal{Q},\mathcal{R}) \text{ for all } m\in\mathbb{N}\cup\{0\}.$$

This gives us, by taking $x=x_{m-1}$, $u=x_m$, $u^*=x_{m+1}$, $y=x_m$ and $v=x_{m+1}$,

$$\begin{aligned}\zeta(G(x_m,x_{m+1},x_{m+1})) \le& \beta(\zeta(M(x_{m-1},x_m,x_m)-d_G(\mathcal{Q},\mathcal{R}))).\zeta[M(x_{m-1},x_m,x_m)\\ &-d_G(\mathcal{Q},\mathcal{R})]+L\zeta[N(x_{m-1},x_m,x_m)-d_G(\mathcal{Q},\mathcal{R})]\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}M(x_{m-1},x_m,x_m) &= \max\{G(x_{m-1},Tx_{m-1},x_m), G(x_{m-1},Tx_{m-1},x_m), G(x_m,Tx_m,x_m),\\ &\quad G(x_m,Tx_m,x_m), G(x_{m-1},x_m,x_m)\}\\ &= \max\{G(x_{m-1},Tx_{m-1},x_m), G(x_m,Tx_m,x_m), G(x_{m-1},x_m,x_m)\}\end{aligned}$$

and

$$\begin{aligned}N(x_{m-1},x_m,x_m) &= \min\{G(x_{m-1},Tx_{m-1},x_m), G(x_m,Tx_m,x_m), G(x_m,Tx_m,x_{m-1}),\\ &\quad G(Tx_{m-1},x_m,x_m)\}\end{aligned}$$

Solving $M(x_{m-1},x_m,x_m)$ by using rectangular inequality and symmetry property of G, we calculate

$$\begin{aligned}G(x_{m-1},Tx_{m-1},x_m) &\le G(x_{m-1},x_m,x_m)+G(x_m,x_m,Tx_{m-1})\\ &\le G(x_{m-1},x_m,x_m)+G(x_m,x_m,Tx_{m-1})+G(Tx_{m-1},Tx_{m-1},x_m)\\ &= G(x_{m-1},x_m,x_m)+d_G(\mathcal{Q},\mathcal{R})\\ G(x_m,Tx_m,x_m) &\le G(Tx_m,x_{m+1},x_{m+1})+G(x_{m+1},x_m,x_m)\\ &\le G(Tx_m,x_{m+1},x_{m+1})+G(x_{m+1},Tx_m,Tx_m)+G(x_{m+1},x_m,x_m)\end{aligned}$$

$$= d_G(\mathcal{Q},\mathcal{R}) + G(x_{m+1},x_m,x_m)$$

implies

$$M(x_{m-1},x_m,x_m) \le \max\{G(x_{m-1},x_m,x_m), G(x_{m+1},x_m,x_m)\} + d_G(\mathcal{Q},\mathcal{R}) \tag{3.3}$$

In a similar manner we solve for $N(x_{m-1},x_m,x_m)$

$$G(Tx_{m-1},x_m,x_m) \le G(Tx_{m-1},x_m,x_m) + G(x_m,Tx_{m-1},Tx_{m-1})$$
$$= d_G(\mathcal{Q},\mathcal{R})$$

imples

$$\begin{aligned} N(x_{m-1},x_m,x_m) \le \min\{&G(x_{m-1},Tx_{m-1},x_m), G(x_m,Tx_m,x_m), G(x_m,Tx_m,x_{m-1}),\\ &d_G(\mathcal{Q},\mathcal{R})\}\\ =\ &d_G(\mathcal{Q},\mathcal{R}) \end{aligned} \tag{3.4}$$

By using equation (3.3), (3.4) in (3.2), we obtain

$$\begin{aligned} \zeta(G(x_m,x_{m+1},x_{m+1})) &\le \beta(\zeta(M(x_{m-1},x_m,x_m) - d_G(\mathcal{Q},\mathcal{R}))).\zeta[\max\{G(x_{m-1},x_m,x_m),\\ &\qquad G(x_{m+1},x_m,x_m)\}] + L\zeta[0]\\ &= \beta(\zeta(M(x_{m-1},x_m,x_m) - d_G(\mathcal{Q},\mathcal{R}))).\zeta[\max\{G(x_{m-1},x_m,x_m),\\ &\qquad G(x_{m+1},x_m,x_m)\}] \end{aligned} \tag{3.5}$$

If for some m, $\max\{G(x_{m-1},x_m,x_m), G(x_{m+1},x_m,x_m)\} = G(x_{m+1},x_m,x_m)$, (3.2) implies

$$\begin{aligned} \zeta(G(x_m,x_{m+1},x_{m+1})) &\le \beta(\zeta(M(x_{m-1},x_m,x_m) - d_G(\mathcal{Q},\mathcal{R}))).\zeta[G(x_{m+1},x_m,x_m)]\\ &< \zeta[G(x_{m+1},x_m,x_m)] \end{aligned}$$

which is contradiction. Therefore, we must have

$$\begin{aligned} M(x_{m-1},x_m,x_m) &\le \max\{G(x_{m-1},x_m,x_m), G(x_{m+1},x_m,x_m)\} + d_G(\mathcal{Q},\mathcal{R})\\ &\le G(x_{m-1},x_m,x_m) + d_G(\mathcal{Q},\mathcal{R}) \end{aligned}$$

for all $m \in \mathbb{N}$. From the equation (3.5), we find that

$$\begin{aligned} \zeta(G(x_m,x_{m+1},x_{m+1})) &\le \beta(\zeta(G(x_{m-1},x_m,x_m))).\zeta(G(x_{m-1},x_m,x_m)) \\ &< \zeta(G(x_{m-1},x_m,x_m)) \end{aligned} \tag{3.6}$$

holds for all $m \in \mathbb{N}$. Since ζ is nondecreasing, then $G(x_m,x_{m+1},x_{m+1}) < G(x_{m-1},x_m,x_m)$ for all m. Consequently, the sequence $\{G(x_m,x_{m+1},x_{m+1})\}$ is decreasing and is bounded below and $\lim_{m\to\infty} G(x_m,x_{m+1},x_{m+1})$ exists. After rewriting (3.6), we get

$$\frac{\zeta(G(x_m,x_{m+1},x_{m+1}))}{\zeta(G(x_{m-1},x_m,x_m))} \le \beta(\zeta(G(x_{m-1},x_m,x_m))) \le 1$$

for each $n \geq 1$. Taking the limit $m \to \infty$, we find

$$\lim_{m\to\infty} \beta(\zeta(G(x_{m-1}, x_m, x_m))) = 1$$

Now, as $\beta \in \Upsilon$, we get $\lim_{m\to\infty} \zeta(G(x_{m-1}, x_m, x_m)) = 0$, that is

$$\lim_{m\to\infty} G(x_{m-1}, x_m, x_m) = 0 \tag{3.7}$$

Now, we prove that $\{x_m\}$ is G-cauchy sequence. On contrary, we assume that $\{G\}$ is not G-cauchy. Thus, there exists an $\epsilon > 0$ for which we can find a sequence $\{x_{m(\iota)}\}, \{x_{l(\iota)}\}$ of $\{x_m\}$ with $l(\iota) > m(\iota) \geq \iota$ such that

$$G(x_{l(\iota)}, x_{l(\iota)+1}, x_{m(\iota)}) \geq \epsilon$$

and

$$G(x_{l(\iota)}, x_{l(\iota)+1}, x_{m(\iota)-1})t < \epsilon \tag{3.8}$$

From proposition (2.1), lemma (2.1) and (G5), we obtain

$$\begin{aligned}\epsilon \leq G(x_{l(\iota)}, x_{l(\iota)+1}, x_{m(\iota)}) &= G(x_{m(\iota)}, x_{l(\iota)}, x_{l(\iota)+1}) \\ &\leq G(x_{m(\iota)}, x_{m(\iota)-1}, x_{m(\iota)-1}) \\ &+ G(x_{m(\iota)-1}, x_{l(\iota)}, x_{l(\iota)+1}) \\ &< \epsilon + G(x_{m(\iota)}, x_{m(\iota)-1}, x_{m(\iota)-1}) \\ &\leq \epsilon + G(x_{m(\iota)-1}, x_{m(\iota)}, x_{m(\iota)})\end{aligned}$$

implies

$$\lim_{\iota\to\infty} G(x_{l(\iota)}, x_{l(\iota)+1}, x_{m(\iota)}) = \epsilon. \tag{3.9}$$

Consider (3.1) with $u = x_{l(\iota)}, u^* = x_{l(\iota)+1}, x = x_{l(\iota)-1}, y = x_{m(\iota)-1}$ and $v = x_{m(\iota)}$, then

$$\begin{aligned}\zeta(G(x_{l(\iota)}, x_{l(\iota)+1}, x_{m(\iota)})) \leq \beta(\zeta(M(x_{l(\iota)-1}, x_{l(\iota)}, x_{m(\iota)-1}) - d_G(\mathcal{Q}, \mathcal{R}))). \\ \zeta[M(x_{l(\iota)-1}, x_{l(\iota)}, x_{m(\iota)-1} - d_G(\mathcal{Q}, \mathcal{R})] \\ + L\zeta[N(x_{l(\iota)-1}, x_{l(\iota)}, x_{m(\iota)-1}) - d_G(\mathcal{Q}, \mathcal{R})]\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}M(x_{l(\iota)-1}, x_{l(\iota)}, x_{m(\iota)-1}) = \max\{&G(x_{l(\iota)-1}, Tx_{l(\iota)-1}, x_{l(\iota)}), G(x_{l(\iota)-1}, Tx_{l(\iota)-1}, x_{m(\iota)-1}), \\ &G(x_{l(\iota)}, Tx_{l(\iota)}, x_{m(\iota)-1}), G(x_{m(\iota)-1}, Tx_{m(\iota)-1}, x_{l(\iota)}), \\ &G(x_{l(\iota)-1}, x_{l(\iota)}, x_{m(\iota)-1})\}\end{aligned}$$

and

$$\begin{aligned}N(x_{l(\iota)-1}, x_{l(\iota)}, x_{m(\iota)-1}) = \min\{&G(x_{l(\iota)-1}, Tx_{l(\iota)-1}, x_{l(\iota)}), G(x_{l(\iota)}, Tx_{l(\iota)}, x_{m(\iota)-1}), \\ &G(x_{l(\iota)}, Tx_{m(\iota)-1}, x_{m(\iota)-1}), G(x_{l(\iota)}, Tx_{l(\iota)-1}, x_{m(\iota)-1})\}\end{aligned} \tag{3.11}$$

Before solving particular terms of $M(x_{l(\iota)-1}, x_{l(\iota)}, x_{m(\iota)-1})$ and $N(x_{l(\iota)-1}, x_{l(\iota)}, x_{m(\iota)-1})$, we solve for $G(x_{l(\iota)}, x_{l(\iota)+1}, x_{m(\iota)})$ by using proposition (2.1), lemma (2.1) and (G5), that is

$$\begin{aligned}
G(x_{l(\iota)}, x_{l(\iota)+1}, x_{m(\iota)}) &\leq G(x_{l(\iota)}, x_{l(\iota)-1}, x_{l(\iota)-1}) + G(x_{l(\iota)-1}, x_{l(\iota)+1}, x_{m(\iota)}) \\
&\leq G(x_{l(\iota)}, x_{l(\iota)-1}, x_{l(\iota)-1}) + G(x_{m(\iota)}, x_{m(\iota)-1}, x_{m(\iota)-1}) \\
&\quad + G(x_{m(\iota)-1}, x_{l(\iota)-1}, x_{l(\iota)+1}) \\
&\leq 2G(x_{l(\iota)-1}, x_{l(\iota)}, x_{l(\iota)}) + 2G(x_{m(\iota)-1}, x_{m(\iota)}, x_{m(\iota)}) \\
&\quad + G(x_{m(\iota)-1}, x_{l(\iota)-1}, x_{l(\iota)+1})
\end{aligned}$$

Taking the limit $\iota \to \infty$ and from equations (3.7) and (3.9), we get

$$\lim_{\iota\to\infty} G(x_{m(\iota)-1}, x_{l(\iota)-1}, x_{l(\iota)+1}) = \epsilon \tag{3.12}$$

Again, by proposition (2.1) and (G5), we have

$$G(x_{m(\iota)-1}, x_{l(\iota)-1}, x_{l(\iota)+1}) \leq G(x_{l(\iota)+1}, x_{l(\iota)}, x_{l(\iota)}) + G(x_{l(\iota)}, x_{l(\iota)-1}, x_{m(\iota)-1})$$

From equation (3.12) and limit $\iota \to \infty$, we find

$$\lim_{\iota\to\infty} G(x_{l(\iota)-1}, x_{l(\iota)}, x_{m(\iota)-1}) = \epsilon. \tag{3.13}$$

From $M(x_{l(\iota)-1}, x_{l(\iota)}, x_{m(\iota)-1})$, we solve

$$\begin{aligned}
G(x_{l(\iota)-1}, Tx_{l(\iota)-1}, x_{l(\iota)}) &\leq G(Tx_{l(\iota)-1}, x_{l(\iota)}, x_{l(\iota)}) + G(x_{l(\iota)}, x_{l(\iota)}, x_{l(\iota)-1}) \\
&\leq G(Tx_{l(\iota)-1}, x_{l(\iota)}, x_{l(\iota)}) + G(x_{l(\iota)}, Tx_{l(\iota)-1}, Tx_{l(\iota)-1}) \\
&+ G(x_{l(\iota)}, x_{l(\iota)}, x_{l(\iota)-1}) \\
&= d_G(\mathcal{Q}, \mathcal{R}) + G(x_{l(\iota)-1}, x_{l(\iota)}, x_{l(\iota)}) && (3.14) \\
G(x_{l(\iota)-1}, Tx_{l(\iota)-1}, x_{m(\iota)-1}) &\leq G(Tx_{l(\iota)-1}, x_{l(\iota)}, x_{l(\iota)}) + G(x_{l(\iota)}, x_{l(\iota)-1}, x_{m(\iota)-1}) \\
&\leq G(Tx_{l(\iota)-1}, x_{l(\iota)}, x_{l(\iota)}) + G(x_{l(\iota)}, Tx_{l(\iota)-1}, x_{l(\iota)-1}) \\
&+ G(x_{l(\iota)}, x_{l(\iota)-1}, x_{m(\iota)-1}) \\
&= d_G(\mathcal{Q}, \mathcal{R}) + G(x_{l(\iota)}, x_{l(\iota)-1}, x_{m(\iota)-1}) && (3.15) \\
G(x_{l(\iota)}, Tx_{l(\iota)}, x_{m(\iota)-1}) &\leq G(Tx_{l(\iota)}, x_{l(\iota)+1}, x_{l(\iota)+1}) + G(x_{l(\iota)+1}, x_{l(\iota)}, x_{m(\iota)-1}) \\
&\leq G(Tx_{l(\iota)}, x_{l(\iota)+1}, x_{l(\iota)+1}) + G(x_{l(\iota)+1}, x_{l(\iota)}, x_{m(\iota)-1}) \\
&+ G(x_{l(\iota)+1}, x_{l(\iota)}, x_{m(\iota)-1}) \\
&= d_G(\mathcal{Q}, \mathcal{R}) + G(x_{l(\iota)+1}, x_{l(\iota)}, x_{m(\iota)-1}) && (3.16) \\
G(x_{m(\iota)-1}, Tx_{m(\iota)-1}, x_{l(\iota)}) &\leq G(Tx_{m(\iota)-1}, x_{m(\iota)}, x_{m(\iota)}) + G(x_{m(\iota)}, x_{m(\iota)-1}, x_{l(\iota)}) \\
&\leq G(Tx_{m(\iota)-1}, x_{m(\iota)}, x_{m(\iota)}) + G(x_{m(\iota)}, Tx_{m(\iota)-1}, Tx_{m(\iota)-1}) \\
&+ G(x_{m(\iota)}, x_{m(\iota)-1}, x_{l(\iota)}) \\
&= d_G(\mathcal{Q}, \mathcal{R}) + G(x_{m(\iota)}, x_{m(\iota)-1}, x_{l(\iota)}) && (3.17)
\end{aligned}$$

Similarly, we solve for $N(x_{l(\iota)-1}, x_{l(\iota)}, x_{m(\iota)-1})$, use (3.14) to (3.17) in (3.11), we get

$$\begin{aligned} M(x_{l(\iota)-1}, x_{l(\iota)}, x_{m(\iota)-1}) \leq \max\{&G(x_{l(\iota)-1}, x_{l(\iota)}, x_{l(\iota)}), G(x_{l(\iota)}, x_{l(\iota)-1}, x_{m(\iota)-1}) \\ &G(x_{l(\iota)+1}, x_{l(\iota)}, x_{m(\iota)-1}), G(x_{m(\iota)}, x_{m(\iota)-1}, x_{l(\iota)}), \\ &G(x_{l(\iota)-1}, x_{l(\iota)}, x_{m(\iota)-1})\} + d_G(\mathcal{Q}, \mathcal{R}) \\ = \max\{&G(x_{l(\iota)-1}, x_{l(\iota)}, x_{l(\iota)}), G(x_{l(\iota)}, x_{l(\iota)-1}, x_{m(\iota)-1}), \\ &G(x_{l(\iota)+1}, x_{l(\iota)}, x_{m(\iota)-1})\} + d_G(\mathcal{Q}, \mathcal{R}) \end{aligned}$$

Taking limit $\iota \to \infty$ on both side, we get

$$\begin{aligned} \lim_{\iota\to\infty} M(x_{l(\iota)-1}, x_{l(\iota)}, x_{m(\iota)-1}) &= \lim_{\iota\to\infty} \max\{G(x_{l(\iota)-1}, x_{l(\iota)}, x_{l(\iota)}), G(x_{l(\iota)}, x_{l(\iota)-1}, x_{m(\iota)-1}), \\ &\quad G(x_{l(\iota)+1}, x_{l(\iota)}, x_{m(\iota)-1})\} + d_G(\mathcal{Q}, \mathcal{R})\} \\ &= \max\{\lim_{\iota\to\infty} G(x_{l(\iota)-1}, x_{l(\iota)}, x_{l(\iota)}), \lim_{\iota\to\infty} G(x_{l(\iota)}, x_{l(\iota)-1}, x_{m(\iota)-1}), \\ &\quad \lim_{\iota\to\infty} G(x_{l(\iota)+1}, x_{l(\iota)}, x_{m(\iota)-1})\} + d_G(\mathcal{Q}, \mathcal{R})\} \\ &= \max\{0, \epsilon, \epsilon, \epsilon, \epsilon\} + d_G(\mathcal{Q}, \mathcal{R}) \\ &= \epsilon + d_G(\mathcal{Q}, \mathcal{R}) \end{aligned}$$

Thus,

$$\lim_{\iota\to\infty} M(x_{l(\iota)-1}, x_{l(\iota)}, x_{m(\iota)-1}) - d_G(\mathcal{Q}, \mathcal{R}) = \epsilon. \tag{3.18}$$

Similarly,

$$\begin{aligned} N(x_{l(\iota)-1}, x_{l(\iota)}, x_{m(\iota)-1}) = \min\{&d_G(\mathcal{Q}, \mathcal{R}) + G(x_{l(\iota)-1}, x_{l(\iota)}, x_{l(\iota)}), \\ &d_G(\mathcal{Q}, \mathcal{R}) + G(x_{l(\iota)+1}, x_{l(\iota)}, x_{m(\iota)-1}), \\ &d_G(\mathcal{Q}, \mathcal{R}) + G(x_{m(\iota)}, x_{l(\iota)}, Tx_{m(\iota)-1}), \\ &d_G(\mathcal{Q}, \mathcal{R}) + G(Tx_{l(\iota)}, x_{l(\iota)}, x_{m(\iota)-1})\} \end{aligned}$$

Taking limit $\iota \to \infty$, we get

$$\begin{aligned} \lim_{\iota\to\infty} N(x_{l(\iota)-1}, x_{l(\iota)}, x_{m(\iota)-1}) = \min\{&d_G(\mathcal{Q}, \mathcal{R}), d_G(\mathcal{Q}, \mathcal{R}) + \epsilon, \\ &d_G(\mathcal{Q}, \mathcal{R}) + \lim_{\iota\to\infty} G(x_{m(\iota)}, x_{l(\iota)}, Tx_{m(\iota)-1}), \\ &d_G(\mathcal{Q}, \mathcal{R}) + \lim_{\iota\to\infty} G(Tx_{l(\iota)}, x_{l(\iota)}, x_{m(\iota)-1})\} \\ = d_G(\mathcal{Q}, \mathcal{R})& \end{aligned}$$

Thus,

$$\lim_{\iota\to\infty} N(x_{l(\iota)-1}, x_{l(\iota)}, x_{m(\iota)-1}) - d_G(\mathcal{Q}, \mathcal{R}) = 0 \tag{3.19}$$

Now, taking the limit $\iota \to \infty$ in (3.10) and using (3.18) and (3.19), we obtain

$$\zeta(\epsilon) \leq \beta(\zeta(\epsilon)).\zeta(\epsilon) + L\zeta(0)$$

$$\zeta(\epsilon) = 1$$

which implies $\epsilon = 0$, which is contradiction. Hence

$$\lim_{l,m\to\infty} \mathrm{G}(x_{l(\iota)}, x_{l(\iota)+1}, x_{m(\iota)}) = 0.$$

Thus, $\{x_m\}$ is G-cauchy sequence. Since $(\mathcal{Q}, \mathrm{G})$ is "complete G-metric space", so there exist $q \in \mathcal{Q}$ such that $x_m \to q$ as $m \to \infty$. From other side , for all $m \in \mathbb{N}$, we can write

$$\begin{aligned} d_{\mathrm{G}}(\mathcal{Q}, \mathcal{R}) &\le d_{\mathrm{G}}(q, Tx_m) \\ &\le d_{\mathrm{G}}(q, x_{m+1}) + d_{\mathrm{G}}(x_{m+1}, Tx_m) \\ &+ d_{\mathrm{G}}(q, x_{m+1}) + d_{\mathrm{G}}(\mathcal{Q}, \mathcal{R}) \end{aligned} \tag{3.20}$$

Taking the limit $m \to \infty$ in (3.20), we have

$$\lim_{m\to\infty} d_{\mathrm{G}}(q, Tx_m) = d_{\mathrm{G}}(q, \mathcal{Q}) = d_{\mathrm{G}}(\mathcal{Q}, \mathcal{R}).$$

Since $\mathcal{R}$ is approximatively compact with respect to $\mathcal{Q}$, so the sequence $\{Tx_m\}$ has a subsequence $\{Tx_{m_{(\iota)}}\}$ that converges to some $r^* \in \mathcal{R}$. Hence

$$d_{\mathrm{G}}(q, r^*) = \lim_{m\to\infty} d_{\mathrm{G}}(x_{m_{(\iota)}+1}, Tx_{m_{(\iota)}}) = d_{\mathrm{G}}(\mathcal{Q}, \mathcal{R}) \tag{3.21}$$

and so $q \in \mathcal{Q}_0$. Now, since $Tq \in T(\mathcal{Q}_0) \subseteq \mathcal{R}_0$, there exists $q^* \in \mathcal{Q}_0$ such that $d_{\mathrm{G}}(q^*, Tq) = d_{\mathrm{G}}(\mathcal{Q}, \mathcal{R})$. Now, from (3.1) with $a = x_m, u = x_{m+1}, u^* = x_{m+2}, c = q, v = q^*$, we have

$$\begin{aligned} \zeta(\mathrm{G}(x_{m+1}, x_{m+2}, q^*)) \le \beta(\zeta(M(x_m, x_{m+1}, q) - d_{\mathrm{G}}(\mathcal{Q}, \mathcal{R}))).\zeta[M(x_m, x_{m+1}, q) \\ - d_{\mathrm{G}}(\mathcal{Q}, \mathcal{R})] + L\zeta[N(x_m, x_{m+1}, q) - d_{\mathrm{G}}(\mathcal{Q}, \mathcal{R})] \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} M(x_m, x_{m+1}, q) =& \max\{\mathrm{G}(x_m, Tx_m, x_{m+1}), \mathrm{G}(x_m, Tx_m, q), \mathrm{G}(x_{m+1}, Tx_{m+1}, q), \\ & \mathrm{G}(q, Tq, x_{m+1}), \mathrm{G}(x_m, x_{m+1}, q)\} \\ \le& \max\{\mathrm{G}(x_m, x_{m+1}, x_{m+1}), \mathrm{G}(x_{m+1}, x_m, p), \mathrm{G}(x_{m+2}, x_{m+1}, p), \\ & \mathrm{G}(q^*, q, x_m)\} + d_{\mathrm{G}}(\mathcal{Q}, \mathcal{R}) \end{aligned} \tag{3.23}$$

$$\begin{aligned} N(x_m, x_{m+1}, q) =& \min\{\mathrm{G}(x_m, Tx_m, x_{m+1}), \mathrm{G}(x_{m+1}, Tx_{m+1}, q), \mathrm{G}(q, Tq, x_m), \\ & \mathrm{G}(Tx_m, x_{m+1}, q)\} \\ & \le \min\{\mathrm{G}(x_m, x_{m+1}, x_{m+1}), \mathrm{G}(x_{m+2}, x_{m+1}, q), \mathrm{G}(q^*, q, x_m), \\ & \mathrm{G}(x_{m+1}, x_{m+1}.p)\} \\ & + d_{\mathrm{G}}(\mathcal{Q}, \mathcal{R}) \end{aligned} \tag{3.24}$$

Taking the limit $m \to \infty$ in (3.23) and (3.24), we obtain

$$\lim_{m\to\infty} M(x_m, x_{m+1}, q) = G(q^*, q, q) + d_G(\mathcal{Q}, \mathcal{R})$$
$$\lim_{m\to\infty} M(x_m, x_{m+1}, q) - d_G(\mathcal{Q}, \mathcal{R}) = G(q^*, q, q) \tag{3.25}$$

and

$$\lim_{m\to\infty} N(x_m, x_{m+1}, q) = d_G(\mathcal{Q}, \mathcal{R})$$
$$\lim_{m\to\infty} N(x_m, x_{m+1}, q) - d_G(\mathcal{Q}, \mathcal{R}) = 0 \tag{3.26}$$

Now taking the limit $m \to \infty$ in (3.22) and using (3.25) and (3.26), we get

$$\zeta(G(q, q, q^*)) \leq \lim_{m\to\infty} \beta(\zeta(M(x_m, x_{m+1}, q) - d_G(\mathcal{Q}, \mathcal{R}))).\zeta(G(q, q, q^*))$$
$$\implies \lim_{m\to\infty} \beta(\zeta(M(x_m, x_{m+1}, q) - d_G(\mathcal{Q}, \mathcal{R}))) \leq 1$$
$$\implies \lim_{m\to\infty} \zeta(M(x_m, x_{m+1}, q) - d_G(\mathcal{Q}, \mathcal{R})) = 0$$

which implies $G(q, q, q^*) = 0$, that is, $q = q^*$. Thus, $d_G(q, Tq) = d_G(\mathcal{Q}, \mathcal{R})$. Therefore, T has a "best proximity point".

Now we prove the uniqueness of "best proximity point". Suppose that $q \neq r$ such that $d_G(q, Tq) = d_G(\mathcal{Q}, \mathcal{R})$ and $d_G(r, Tr) = d_G(\mathcal{Q}, \mathcal{R})$.

From (3.1) with $x = u = u^* = q$ and $y = v = r$, we get

$$\zeta(G(q, q, r)) \leq \beta(\zeta(M(q, q, r) - d_G(\mathcal{Q}, \mathcal{R}))).\zeta[M(q, q, r) - d_G(\mathcal{Q}, \mathcal{R})]$$
$$+ L\zeta[N(q, q, r) - d_G(\mathcal{Q}, \mathcal{R})] \tag{3.27}$$

where

$$\begin{aligned} M(q, q, r) &= \max\{G(q, Tq, q), G(q, Tq, r), G(q, Tq, r), G(r, Tr, q), G(q, q, r)\} \\ &\leq \max\{G(q, Tq, q), G(q, Tq, r), G(r, Tr, q), G(q, q, r)\} \\ &\leq \max\{d_G(\mathcal{Q}, \mathcal{R}), d_G(\mathcal{Q}, \mathcal{R}) + G(q, q, r), d_G(\mathcal{Q}, \mathcal{R}) + G(r, r, q), G(q, q, r)\} \\ &= \max\{G(q, q, r), G(r, r, q)\} + d_G(\mathcal{Q}, \mathcal{R}) \end{aligned}$$

and

$$\begin{aligned} N(q, q, r) &= \min\{G(q, Tq, q), G(q, Tq, r), G(r, Tr, q), G(Tq, q, r)\} \\ &\leq \min\{d_G(\mathcal{Q}, \mathcal{R}), d_G(\mathcal{Q}, \mathcal{R}) + G(q, q, r), d_G(\mathcal{Q}, \mathcal{R}) + G(r, r, q)\} \\ &= d_G(\mathcal{Q}, \mathcal{R}) \end{aligned}$$

If $\max\{G(q, q, r), G(r, r, q)\} = G(r, r, q)$ then from (3.27), we get

$$\begin{aligned} \zeta(G(q, q, r)) &\leq \beta(\zeta(M(q, q, r) - d_G(\mathcal{Q}, \mathcal{R}))).\zeta(G(q, q, r)) \\ &< \zeta(G(q, q, r)) \end{aligned}$$

which is contradiction. Thus $\max\{G(q,q,r), G(r,r,q)\} = G(r,r,q)$, again (3.27) implies

$$\zeta(G(q,q,r)) \leq \beta(\zeta(M(q,q,r) - d_G(\mathcal{Q},\mathcal{R}))).\zeta(G(r,r,q)$$
$$< \zeta(G(r,r,q)$$

As ζ is non decreasing, then $r = q$. Thus, the result. □

Example 3.1. *Let $X = [0,\infty)$ and*

$$G(x,y,z) = \frac{1}{4}\{|x-y| + |y-z| + |z-x|\}$$

be G-metric on X defined by $d_G(x,y) = |x-y|$. Let "$\mathcal{Q} = \{3,4,5,6,7\}$" and "$\mathcal{R} = \{9,10,11,12,13\}$". Define $T : \mathcal{Q} \to \mathcal{R}$ by

$$T(x) = \begin{cases} 9, & if x = 7 \\ x+6, & otherwise \end{cases}$$

Also, consider $\zeta : [0,\infty) \to [0,\infty)$ and $\beta : [0,\infty) \to [0,1)$ defined by $\zeta(x) = \frac{x}{2}, \beta(x) = \frac{x}{(1+x)}$ respectively. Clearly, here $d_G(\mathcal{Q},\mathcal{R}) = 2, \mathcal{Q}_0 = \{7\}, \mathcal{R}_0 = \{9\}$ and $T\mathcal{Q}_0 \subseteq \mathcal{R}_0$. Let $d_G(u,Tx) = d_G(\mathcal{Q},\mathcal{R})$ and $d_G(v,Ty) = d_G(\mathcal{Q},\mathcal{R})$, then $(u,x),(v,y) \in \{(7,7),(7,3)\}$. Also if $d_G(u^*,Tu) = d_G(\mathcal{Q},\mathcal{R}) = 2$, then $u^* = 7$. Therefore, if

$$d_G(u,Tx) = d_G(\mathcal{Q},\mathcal{R})$$
$$d_G(u^*,Tu) = d_G(\mathcal{Q},\mathcal{R})$$
$$d_G(v,Ty) = d_G(\mathcal{Q},\mathcal{R})$$

then

$$(u,u^*,v,x,y) \in \{(7,7,7,7,7),(7,7,7,3,3),(7,7,7,3,7),(7,7,7,7,3)\}$$

from which we get

$$M(x,u,y) = N(x,u,y) = 9$$

Now, as $u = u^* = v = 7$, so $\zeta(G(u,u^*,v)) = 0$. Hence,

$$\begin{aligned}
\zeta(G(u,u^*,v)) = 0 &\leq \frac{1}{2}x \leq \frac{1}{2}(Tx-2)\\
&\leq \frac{1}{2}(Tx-2)\{-1+\frac{L}{2}(Ty-2)\}\\
&< \frac{1}{2}(Tx-2)\{\frac{1}{2}(Tx-2) - \frac{1}{2}Tx + \frac{L}{2}(Ty-2)\} + \frac{L}{2}(Ty-2)\\
&< \frac{1}{2}(Tx-2)\{\frac{1}{2}(Tx-2) + L.\frac{1}{2}(Tx-2) - \frac{1}{2}x\} + L.\frac{1}{2}(Ty-2)\\
&= \zeta(M(x,u,y) - d_G(\mathcal{Q},\mathcal{R}))[\zeta(M(x,u,y) - d_G(\mathcal{Q},\mathcal{R}))\\
&\quad + L.\zeta(N(x,u,y) - d_G(\mathcal{Q},\mathcal{R})) - \zeta(G(u,u^*,v))] + L.\zeta(N(x,u,y)\\
&\quad - d_G(\mathcal{Q},\mathcal{R}))
\end{aligned}$$

$$\zeta(G(u,u^*,v)) \leq \zeta(M(x,u,y) - d_G(\mathcal{Q},\mathcal{R})).\zeta(M(x,u,y) - d_G(\mathcal{Q},\mathcal{R}))$$

$$
\begin{aligned}
&\quad + L.\zeta(N(x,u,y) - d_{\mathsf{G}}(\mathcal{Q},\mathcal{R}))[1 + \zeta(M(x,u,y) - d_{\mathsf{G}}(\mathcal{Q},\mathcal{R}))] \\
&\quad - \zeta(M(x,u,y) - d_{\mathsf{G}}(\mathcal{Q},\mathcal{R})).\zeta(\mathsf{G}(u,u^*,v)) \\
\zeta(\mathsf{G}(u,u^*,v))[1 + \zeta(M(x,u,y) - d_{\mathsf{G}}(\mathcal{Q},\mathcal{R}))] & \\
&\leq \zeta(M(x,u,y) - d_{\mathsf{G}}(\mathcal{Q},\mathcal{R})).\zeta(M(x,u,y) - d_{\mathsf{G}}(\mathcal{Q},\mathcal{R})) \\
&\quad + L.\zeta(N(x,u,y) - d_{\mathsf{G}}(\mathcal{Q},\mathcal{R})) \\
\zeta(\mathsf{G}(u,u^*,v)) &\leq \frac{\zeta(M(x,u,y) - d_{\mathsf{G}}(\mathcal{Q},\mathcal{R}))}{1 + \zeta(M(x,u,y) - d_{\mathsf{G}}(\mathcal{Q},\mathcal{R}))}.\zeta(M(x,u,y) - d_{\mathsf{G}}(\mathcal{Q},\mathcal{R})) \\
&\quad + L.\zeta(N(x,u,y) - d_{\mathsf{G}}(\mathcal{Q},\mathcal{R})) \\
&\leq \beta(\zeta(M(x,u,y) - d_{\mathsf{G}}(\mathcal{Q},\mathcal{R}))).\zeta(M(x,u,y) - d_{\mathsf{G}}(\mathcal{Q},\mathcal{R})) \\
&\quad + L.\zeta(N(x,u,y) - d_{\mathsf{G}}(\mathcal{Q},\mathcal{R}))
\end{aligned}
$$

Thus, T is G*-generalized $\zeta - \beta - T$ contraction mapping and all the conditions of thereom are satisfied with $q = 7$ as unique "best proximity point".*

If in theorem (3.1), $\zeta(x) = x$, then we obtain the following corollary.

Corollary 3.1. *Let $(\mathcal{Q},\mathcal{R})$ be pair of nonempty closed subset of* G*-metric space (X, G) such that $(\mathcal{Q}, \mathsf{G})$ is "complete* G*-metric space" and $\mathcal{R}$ is approximatively compact w.r.t. $\mathcal{Q}$. Consider $T : \mathcal{Q} \to \mathcal{R}$ be non self mapping satisfying $T(\mathcal{Q}_0) \subseteq \mathcal{R}_0$ and for $x, y, u, u^*, v \in \mathcal{Q}$ and $L \geq 1$, defined by*

$$
\begin{aligned}
d_{\mathsf{G}}(u, Tx) &= d_{\mathsf{G}}(\mathcal{Q},\mathcal{R}) \\
d_{\mathsf{G}}(u^*, Tu) &= d_{\mathsf{G}}(\mathcal{Q},\mathcal{R}) \\
d_{\mathsf{G}}(v, Ty) &= d_{\mathsf{G}}(\mathcal{Q},\mathcal{R}) \\
\implies \mathsf{G}(u,u^*,v) &\leq \beta(M(x,u,y) - d_{\mathsf{G}}(\mathcal{Q},\mathcal{R})).(M(x,u,y) - d_{\mathsf{G}}(\mathcal{Q},\mathcal{R})) \\
&\quad + L\zeta[N(x,u,y) - d_{\mathsf{G}}(\mathcal{Q},\mathcal{R})]
\end{aligned}
$$

where $M(x,u,y) = \max\{\mathsf{G}(x,Tx,u), \mathsf{G}(x,Tx,y), \mathsf{G}(u,Tu,y), \mathsf{G}(y,Ty,u), \mathsf{G}(x,u,y)\}$ and $N(x,u,y) = \min\{\mathsf{G}(x,Tx,u), \mathsf{G}(u,Tu,y), \mathsf{G}(y,Ty,x), \mathsf{G}(Tx,u,y)\}$ Then T has a unique "best proximity point" in $\mathcal{Q}$.

If we proceed with the above corollary by considering $\beta(x) = s$ where $0 \leq s < 1$, then we get another corollary as defined below.

Corollary 3.2. *Let $(\mathcal{Q},\mathcal{R})$ be pair of nonempty closed subset of* G*-metric space (X, G) such that $(\mathcal{Q}, \mathsf{G})$ is "complete* G*-metric space" and $\mathcal{R}$ is approximately compact w.r.t. $\mathcal{Q}$. Consider $T : \mathcal{Q} \to \mathcal{R}$ be non self mapping satisfying $T(\mathcal{Q}_0) \subseteq \mathcal{R}_0$ and for $x, y, u, u^*, v \in \mathcal{Q}$ and $L \geq 1$, defined by*

$$d_{\mathsf{G}}(u, Tx) = d_{\mathsf{G}}(\mathcal{Q},\mathcal{R})$$

$$d_G(u^*, Tu) = d_G(\mathcal{Q}, \mathcal{R})$$

$$d_G(v, Ty) = d_G(\mathcal{Q}, \mathcal{R})$$

$$\implies \zeta(G(u, u^*, v)) \leq s.(M(x, u, y) - d_G(\mathcal{Q}, \mathcal{R})) + L\zeta[N(x, u, y) - d_G(\mathcal{Q}, \mathcal{R})]$$

where $M(a, u, y) = \max\{G(x, Tx, u), G(x, Tx, y), G(u, Tu, y), G(y, Ty, u), G(x, u, y)\}$ *xnd* $N(x, u, y) = \min\{G(x, Tx, u), G(u, Tu, y), G(y, Ty, x), G(Tx, u, y)\}$ *Then* T *has a unique "best proximity point" in* $\mathcal{Q}$*.*

Remark 3.1. *The "best proximity point" theorem (3.1) is reduced to the result of [4], if "complete* G*-metric spaces" becomes complete Metric spaces.*

4. Application to Fixed Point Theory

In this section, we discuss the fixed point theorem as an application part of "best proximity point" theorem. By considering $\mathcal{Q} = \mathcal{R} = X$, in

$$d_G(u, Tx) = d_G(\mathcal{Q}, \mathcal{R})$$

$$d_G(u^*, Tu) = d_G(\mathcal{Q}, \mathcal{R})$$

$$d_G(v, Ty) = d_G(\mathcal{Q}, \mathcal{R})$$

we get, $u = Tx, u^* = Tu = T^2x$ and $v = Ty$. Therefore, theorem (3.1) restates as:

Theorem 4.1. *Let* (X, G) *be "complete* G*-metric space". Consider* $\mathcal{Q}$ *as a nonempty subset of* X*. Let* $T : \mathcal{Q} \to \mathcal{Q}$ *be mapping satisfying the successive condition*

$$\zeta(G(Tx, T^2x, Tc)) \leq \beta(\zeta(M(x, Tx, y))).\zeta(M(x, Tx, y)) + L\zeta[N(x, Tx, y)]$$

where $\zeta \in \Xi, \beta \in \Upsilon, L \geq 1,$
$M(x, Tx, y) = \max\{G(x, Tx, Tx), G(x, Tx, y), G(Tx, T^2x, y), G(y, Ty, Tx),$
$G(x, Tx, y)\}$ *and* $N(x, Tx, y) = \min\{G(x, Tx, Tx), G(Tx, T^2x, y), G(y, Ty, x),$
$G(Tx, Tx, y)\}$
Then T *has a fixed point.*

After taking $\zeta(x) = x$ in theorem (4), we obtain a corollary, stated as:

Corollary 4.1. *Let* (X, G) *be "complete* G*-metric space". Consider* $\mathcal{Q}$ *as a nonempty subset of* X*. Let* $T : \mathcal{Q} \to \mathcal{Q}$ *be mapping satisfying the successive condition*

$$G(Tx, T^2x, Ty) \leq \beta(M(x, Tx, y)).(M(x, Tx, y)) + L\zeta[N(x, Tx, y)]$$

where $\beta \in \Upsilon, L \geq 1,$
$M(x, Tx, y) = \max\{G(x, Tx, Tx), G(x, Tx, y), G(Tx, T^2x, y), G(y, Ty, Tx),$
$G(x, Tx, y)\}$ *and* $N(x, Tx, y) = \min\{G(x, Tx, Tx), G(Tx, T^2x, y), G(y, Ty, x),$

$G(Tx, Tx, y)\}$

Then T has a fixed point.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] A.H. Ansari, A. Razani, N. Hussain, New Best Proximity Point Results in G-Metric Space, J. Linear Topol. Algebra, 6 (2017), 73-89. https://jlta.ctb.iau.ir/article_530221.html

[2] A.H. Ansari, S. Changdok, N. Hussain, et al. Some Common Fixed Point Theorems for Weakly α-Admissible Pairs in G-Metric Spaces With Auxiliary Functions, J. Math. Anal. 8 (2017), 80–107. http://www.ilirias.com/jma/repository/docs/JMA8-3-7.pdf

[3] B.S. Choudhury, P. Maity, Best Proximity Point Results in Generalized Metric Spaces, Vietnam J. Math. 44 (2015), 339–349. https://doi.org/10.1007/s10013-015-0141-3.

[4] H. Aydi, E. Karapınar, İ.M. Erhan, P. Salimi, Best proximity points of generalized almost ζ-Geraghty contractive non-self-mappings, Fixed Point Theory Appl. 2014 (2014), 32. https://doi.org/10.1186/1687-1812-2014-32.

[5] M. Abbas, A. Hussain, P. Kumam, A Coincidence Best Proximity Point Problem in G-Metric Spaces, Abstr. Appl. Anal. 2015 (2015), 243753. https://doi.org/10.1155/2015/243753.

[6] N. Hussain, A. Latif, P. Salimi, Best Proximity Point Results in G-metric Spaces, Abstr. Appl. Anal. 2014 (2014), 837943. https://doi.org/10.1155/2014/837943.

[7] N. Priyobarta, B. Khomdram, Y. Rohen, N. Saleem, On Generalized Rational α-Geraghty Contraction Mappings in G-Metric Spaces, J. Math. 2021 (2021), 6661045. https://doi.org/10.1155/2021/6661045.

[8] Z. Mustafa, A New Structure for Generalized Metric Spaces With Applications to Fixed Point Theory, Ph.D. Thesis, The University of Newcastle, New SouthWales, Australia, 2005.

[9] Z. Mustafa, B. Sims, A New Approach to Generalized Metric Spaces, J. Nonlinear Convex Anal. 7 (2006), 289–297. http://yokohamapublishers.jp/online2/opjnca/vol7/p289.html