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Research article

Algorithm for split variational inequality, split equilibrium problem and split common fixed point problem

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Abstract: The main aim of this manuscript is to work on the split equilibrium problem with the combined results of the fixed point problem and split variational inequality problem. This paper is an extension of the recent work of Lohawech et al. We proposed a sequence that converges weakly to the common solution of all the three problems mentioned earlier. In the end, we supply some direct consequences of the main result, as the paper is an extension of various existing results.

Keywords: variational inequality; equilibrium problem; fixed point; split variational inequality; split equilibrium problem
Mathematics Subject Classification: 58E35, 47H09, 47H10

1. Introduction

Split equilibrium problem (SEP), introduced by He [20] in 2012, which is defined as: find the solution of equilibrium problem (EP) whose image is also a solution of another EP under a given bounded linear operator. Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two real Hilbert spaces and C and Q be closed convex subsets of $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively and $B : \mathbb{H}_1 \rightarrow \mathbb{H}_2$ be a linear and bounded operator. Let f and F be two functions from $C \times C$ and $Q \times Q$ to $\mathbb{R}$, respectively, then SEP is to get a point $c \in C$ such that

$$f(c, c^*) \geq 0 \ \forall \ c^* \in C \tag{1.1}$$

and

$$d = Bc \in Q \text{ solves } F(d, d^*) \geq 0 \ \forall \ d^* \in Q. \tag{1.2}$$

Equation (1.1) is named as classical Equilibrium Problem given by Blum [2] and its solution set is denoted by $EP(f)$.

SEP provides us a way to split the solution between two different subsets such that the solution of one problem and its image implies the solution of another problem under the imposed bounded linear operator. As the special case of SEP, split variational inequality problem (SVIP) was introduced by

Censor et al. [17], in 2012. The SVIP is stated as: consider g and g′ be two operator for $\mathbb{H}_1$ and $\mathbb{H}_2$, two Hilbert Spaces, respectively, $B : \mathbb{H}_1 \to \mathbb{H}_2$ be a linear and bounded operator, C and Q be the same as defined above, then

$$c \in C \text{ such that } \langle g(c), c^* - c\rangle \geq 0 \; \forall \; c^* \in C \tag{1.3}$$

and

$$d = Bc \in Q \text{ such that } \langle g'(d), d^* - d\rangle \geq 0 \; \forall \; d^* \in Q. \tag{1.4}$$

and the equation (1.3) separately gives classical variational inequality (VI).

Split common fixed point problem (SCFPP) , introduced by Censor and Segal [18], in 2009. The SCFPP problem is to find a point $c \in Fix(\mathrm{S})$ gives $Bc \in Fix(\mathrm{T})$, where $B : \mathbb{H}_1 \to \mathbb{H}_2$ is linear and bounded operator with $\mathrm{S} : \mathbb{H}_1 \to \mathbb{H}_1$ and $\mathrm{T} : \mathbb{H}_2 \to \mathbb{H}_2$ are the general operators and $Fix()$ denoted the solution set of fixed point of the considered mapping. SCFPP is the generalization of Split Feasibility Problem, given by Censor and Elfving [19], in 1994 and this problem formulate a point $c \in C$ with $Bc \in Q$ where C and Q are convex subsets of $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively.

Korpelevich [3], in 1976, introduced the extragradient method for solving Eq (1.3) when g is monotone and k-Lipschitz continuous in the finite dimensional Euclidean space . In 2003, Takahashi and Toyoda [13] introduced the following method to calculate the common solution of VIP and fixed point problem (FPP)

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n) S P_C(x_n - \gamma_n f x_n),$$

where $S : C \to C$ is nonexpansive mapping and $f : C \to \mathbb{H}_1$ is a ν-inverse strongly monotone mapping.

After that, in 2006 Nadezhkina and Takahashi [14] suggested the modified extragradient method to prove the weak convergence of the defined iteration to the common solution of VIP and FPP:

$$\begin{aligned} y_n &= P_C(x_n - \gamma_n f x_n), \\ x_{n+1} &= \lambda_n x_n + (1 - \lambda_n) S P_C(x_n - \gamma_n f y_n), \end{aligned}$$

where $S : C \to C$ is nonexpansive mapping and $f : C \to \mathbb{H}_1$ is a monotone and k-Lipschitz continuous mapping. Later on , in 2011, Kangtunyakarn [1] proved the convergence theorem for calculating the common point of the three sets of solutions of equilibrium problem, variational inequality and the fixed point problems by practicing with a newly developed mapping achieved by infinite family of real numbers and of nonexpansive mappings.

In 2017, Tian et al. [11] proposed an algorithm for finding an element to solve the class of SVIP by combining extragradient method with CQ algorithm. In 2018, Lohawech et al. [12] proved the existence of common solution of the SVIP and FPP by the introduced iterative method which was inspired from Nadezhkina and Takahashi's [14] modified extragradient method and Xu's [7] algorithm and proved its weak convergence.

Inspired from work of Tian et al. [11] and Lohewech et al. [12], we extend results in the direction of findings the common solution of SVIP and SEP with SCFPP.

2. Preliminaries

Lemma 2.1. *[9] Given $a \in \mathbb{H}$ and $z \in C$, then the following statements are equivalent:*

(i) $z = P_C a$;

(ii) $\langle a - z, z - b\rangle \geq 0 \ \forall\ b \in C$;

(iii) $\langle a - P_C a, b - P_C a\rangle \leq 0 \ \forall\ a \in \mathbb{H}, b \in C$;

(iv) $\|a - b\|^2 \geq \|a - z\|^2 + \|b - z\|^2 \ \forall\ b \in C$.

Lemma 2.2. *[2] Let the function* $f : C \times C \to \mathbb{R}$ *satisfy the following conditions:*

(i) $f(u, u) = 0 \ \forall\ u \in C$;

(ii) f *is monotone, i.e.* $f(u, v) + f(v, u) \leq 0 \ \forall\ u, v \in C$;

(iii) for each $u, v, w \in C, \lim_{t\to 0} f(tw + (1 - t)u, v) \leq f(u, v)$;

(iv) for each $u \in C, f(u, .)$ *is convex and lower semicontinuous.*
Then $EP(f) \neq \phi$.

Lemma 2.3. *[2] Let* $r > 0, u \in \mathbb{H}$, *and* f *satisfy the conditions (i)-(iv) in Lemma* (2.2), *then there exists* $w \in C$ *such that* $f(w, v) + \frac{1}{r}\langle v - w, w - u\rangle \geq 0 \forall\ v \in C$.

Lemma 2.4. *[2] Let* $r > 0, u \in \mathbb{H}$, *and* f *satisfy the conditions (i)-(iv) in Lemma* (2.2). *Define a mapping* $\mathrm{T}_r : \mathbb{H} \to C$ *as* $\mathrm{T}_r(u) = \{w \in C : f(w, v) + \frac{1}{r}\langle v - w, w - u\rangle \geq 0 \forall\ v \in C\}$. *Then the following hold:*

(i) T_r *is single-valued;*

(ii) T_r *is firmly nonexpansive, i.e.,* $\|\mathrm{T}_r u - \mathrm{T}_r v\| \leq \langle \mathrm{T}_r u - \mathrm{T}_r v, u - v\rangle$ *for all* $u, v \in \mathbb{H}$;

(iii) $EP(f) = F(\mathrm{T}_r)$, *where* $F(\mathrm{T}_r)$ *denotes the sets of fixed point of* T_r;

(iv) $EP(f)$ *is closed and convex.*

Definition 2.1. [15] Let $A : \mathbb{H} \to \mathbb{H}$ be a set valued mapping with the effective domain $D(A) = \{x \in \mathbb{H} : Ax \neq \phi\}$.

The set valued mapping A is said to be monotone if, for each $x, y \in D(A), u \in Ax$ and $v \in Ay$, we have

$$\langle x - y, u - v\rangle \geq 0.$$

As, graph of A is defined by $G(A) = \{(x, y) : y \in Ax\}$ and the mapping A is maximal if its graph $G(A)$ is not appropriately contained in the graph of any other mapping which is of same type as A. The accompanying nature of the maximal monotone mappings is advantageous and supportive to utilize:
A monotone mapping A is maximal if and only if, for $(x, u) \in \mathbb{H} \times \mathbb{H}$,

$$\langle x - y, u - v\rangle \geq 0 \text{ for each } (y, v) \in G(A) \text{ implies } u \in Ax.$$

For maximal monotone set-valued mapping A on $\mathbb{H}$ and $r > 0$, the operator

$$J_r := (I - rA)^{-1} : \mathbb{H} \to D(A)$$

is called the resolvent of A.
Consider $f : C \to \mathbb{H}$ be a monotone and k-Lipschitz continuous mapping . In [5], normal cone to C is specified by

$$N_C x = \{z \in \mathbb{H} : \langle z, y - x\rangle \leq 0,\ \forall\, y \in C\} \text{ for all } x \in C$$

is maximal monotone and resolvent of N_C is P_C.

Lemma 2.5. *[14] Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be real Hilbert spaces. Let $A : \mathbb{H}_1 \to \mathbb{H}_1$ be a maximal monotone mapping and J_r be the resolvent of A for $r < 0$. Suppose that $\mathrm{T} : \mathbb{H}_2 \to \mathbb{H}_2$ is a nonexpansive mapping and $B : \mathbb{H}_1 \to \mathbb{H}_1$ is a bounded linear operator. Assume that $A^{-1}0 \cap B^{-1}Fix(\mathrm{T}) \neq \phi$. Let $r, \alpha > 0$ and $z \in \mathbb{H}_1$. Then the following statements are equivalent:*

(i) $z = J_r(I - \alpha B^*(I - \mathrm{T})B)z$;

(ii) $0 \in B^*(I - T)Bz + Az$;

(iii) $z \in A^{-1}0 \cap B^{-1}Fix(\mathrm{T})$.

Lemma 2.6. *[16] Let $\{\beta_n\}$ be a real sequence satisfying $0 < a \leq \beta_n \leq b < 1$ for all $n \geq 0$, and let $\{\mu_n\}$ and $\{\nu_n\}$ be two sequences in $\mathbb{H}$ such that, for some $\eta \geq 0$,*

$$\lim_{n\to\infty} \sup\|\mu_n\| \leq \eta,$$
$$\lim_{n\to\infty} \sup\|\{\nu_n\} \leq \eta,$$
$$\text{and} \lim_{n\to\infty}\|\beta_n\mu_n + (1-\beta_n)\nu_n\| = \eta.$$

Lemma 2.7. *[6] Let $\{a_n\}$ be a sequence in $\mathbb{H}$ satisfying the properties:*

(i) $\lim_{n\to\infty}\|a_n - a\|$ *exists for each $a \in C$;*

(ii) *$\omega_w(a_n) \subset C$, where $\omega_w(a_n)$ represents the set of all weak cluster points of* $\{a_n\}$.

Then $\{a_n\}$ converges weakly to a point in C.

3. Main results

Throughout, we consider C and Q both are nonempty closed convex subsets of real Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively. Suppose that $B : \mathbb{H}_1 \to \mathbb{H}_2$ is a non-zero bounded linear operator, $f : C \times C \to \mathbb{R}$ and $F : Q \times Q \to \mathbb{R}$ be two functions satisfiy the conditions (i) to (iv) of Lemma (2.2), $\mathrm{g} : C \to \mathbb{H}_1$ is a monotone and k-Lipschitz continuous mapping and $\mathrm{g}' : \mathbb{H}_2 \to \mathbb{H}_2$ is a δ- inverse strongly monotone mapping . Suppose $\mathrm{T} : \mathbb{H}_2 \to \mathbb{H}_2$ and $\mathrm{S} : C \to C$ are nonexpansive mappings . Let $\{\mu_n\}, \{\alpha_n\} \subset (0, 1), \{\gamma_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{\|B\|^2}\right)$ and $\{\lambda_n\} \subset [c, d]$ where $c, d \in (0, \frac{1}{k})$.

Initially, we define an algorithm for solving VIP, SCFPP and SEP, the purpose of which is to discover an element a^* in such a way that

$$a^* \in VI(C, \mathrm{g}) \cap Fix(\mathrm{S}) \cap EP(f) \text{ and } Ba^* \in Fix(\mathrm{T}) \cap EP(F). \tag{3.1}$$

Theorem 3.1. *Fix* $\daleth = \{u \in VI(C, g) \cap Fix(S) \cap EP(f) : Bu \in Fix(T) \cap EP(F)\}$ *and consider that* $\daleth \neq \phi$. *Let the sequences* $\{u_n\}, \{v_n\}, \{a_n\}, \{b_n\}$ *and* $\{c_n\}$ *be defined by* $a_1 = a \in C$ *and*

$$\begin{aligned}
& f(v_n, z) + \frac{1}{r_n}\langle z - v_n, v_n - a_n\rangle \geq 0, \\
& F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - Bv_n\rangle \geq 0, \\
& c_n = \mu_n u_n + (1 - \mu_n) P_C(u_n - \gamma_n B^*(I - T)Bu_n), \\
& b_n = P_C(c_n - \lambda_n g(c_n)), \\
& a_{n+1} = \alpha_n c_n + (1 - \alpha_n) S P_C(c_n - \lambda_n g(b_n)),
\end{aligned} \tag{3.2}$$

for all $n \in \mathbb{N}$. *Then, the sequence* $\{a_n\}$ *weakly converges to an element* $u \in \daleth$, *where* $u = \lim_{n\to\infty} P_\daleth a_n$.

Proof. Let $a \in \daleth$ and consider $\{T_{r_n}\}$ be a sequence of mapping as stated in Lemma (2.4), gives $a = P_C(a - \lambda_n Ba) = T^f_{r_n} a$, also $a = T^F_{r_n} Ba$ as $a \in EP(f)$ and $Ba \in EP(F)$.

From Theorem 3.1 of [11] that $P_C(I - \gamma_n B^*(I - T)B)$ is $\frac{1+\gamma_n\|B\|^2}{2}$ averaged . It is clear to see from Lemma 2.2 of [10] that $\mu_n I + (1-\mu_n)P_C(I - \gamma_n B^*(I - T)B)$ is $\mu_n + (1-\mu_n)\frac{1+\gamma_n\|B\|^2}{2}$ averaged. So, c_n can be written as

$$c_n = (1 - \beta_n)u_n + \beta_n V_n u_n \tag{3.3}$$

where $\beta_n = \mu_n + (1 - \mu_n)\frac{1+\gamma_n\|B\|^2}{2}$ and V_n is a nonexpansive mapping for each $n \in \mathbb{N}$.
Let $a \in \daleth$ and from Lemma (2.4), we obtain

$$\begin{aligned}
\|v_n - a\|^2 &= \|T^f_{r_n} a_n - T^f_{r_n} a\|^2 \\
&\leq \langle T^f_{r_n} a_n - T^f_{r_n} a, a_n - a\rangle \\
&= \langle v_n - a, a_n - a\rangle \\
&= \frac{1}{2}(\|v_n - a\|^2 + \|a_n - a\|^2 - \|a_n - v_n\|^2)
\end{aligned}$$

and hence,

$$\begin{aligned}
\|v_n - a\|^2 &\leq \|a_n - a\|^2 - \|a_n - v_n\|^2 \\
&\leq \|a_n - a\|^2.
\end{aligned} \tag{3.4}$$

Also,

$$\begin{aligned}
\|u_n - a\|^2 &= \|T^F_{r_n} v_n - T^F_{r_n} a\| \\
&\leq \langle u_n - a, v_n - a\rangle \\
&= \frac{1}{2}(\|u_n - a\|^2 + \|v_n - a\|^2 - \|u_n - v_n\|^2), \\
\|u_n - a\|^2 &\leq \|v_n - a\|^2 - \|u_n - v_n\|^2 \\
&\leq \|a_n - a\|^2 - \|u_n - v_n\|^2 \\
&\leq \|a_n - a\|^2.
\end{aligned} \tag{3.5}$$

From (3.5)

$$\begin{aligned}
\|c_n - a\|^2 &\leq \|(1 - \beta_n)(u_n - a) + \beta_n(V_n u_n - a)^2\| \\
&= (1 - \beta_n)\|u_n - a\|^2 + \beta_n\|V_n u_n - a\|^2 - \beta_n(1 - \beta_n)\|u_n - V_n u_n\|^2 \\
&\leq \|u_n - a\|^2 - \beta_n(1 - \beta_n)\|u_n - V_n u_n\|^2 \\
&\leq \|u_n - a\|^2
\end{aligned}$$

$$\leq \|a_n - a\|^2, \tag{3.6}$$

implies

$$\beta_n(1-\beta_n)\|u_n - V_n u_n\|^2 \leq \|a_n - a\|^2 - \|c_n - a\|^2. \tag{3.7}$$

Now, set $z_n = P_C(c_n - \lambda_n \mathrm{g}(b_n))$ for all $n \geq 0$. It pursue from Lemma (2.1) that

$$\begin{aligned}\|z_n - a\|^2 &\leq \|c_n - \lambda_n \mathrm{g}(b_n) - a\|^2 - \|c_n - \lambda_n \mathrm{g}(b_n) - z_n\|^2 \\ &\leq \|c_n - a\|^2 - \|c_n - z_n\|^2 + 2\lambda_n\langle \mathrm{g}(b_n), a - z_n\rangle \\ &\leq \|c_n - a\|^2 - \|c_n - z_n\|^2 + 2\lambda_n\langle \mathrm{g}(b_n) - \mathrm{g}(a), a - b_n\rangle + 2\lambda_n\langle \mathrm{g}(a), a - b_n\rangle + 2\langle \mathrm{g}(b_n), b_n - z_n\rangle.\end{aligned} \tag{3.8}$$

Using the monotonicity of g and a is solution of $VIP(\mathrm{g}, C)$, we have

$$\langle \mathrm{g}(b_n) - \mathrm{g}(a), a - b_n\rangle \leq 0 \text{ and } \langle \mathrm{g}(a), a - b_n\rangle \leq 0. \tag{3.9}$$

From Eqs (3.8) and (3.9), we obtain

$$\begin{aligned}\|z_n - a\|^2 &\leq \|c_n - a\|^2 - \|c_n - z_n\|^2 + 2\lambda_n\langle \mathrm{g}(b_n), b_n - z_n\rangle \\ &= \|c_n - a\|^2 - \|c_n - b_n\|^2 - \|b_n - z_n\|^2 - 2\langle c_n - b_n, b_n - z_n\rangle + 2\lambda_n\langle \mathrm{g}(b_n), b_n - z_n\rangle \\ &= \|c_n - a\|^2 - \|c_n - b_n\|^2 - \|b_n - z_n\|^2 + 2\langle c_n - b_n - \lambda_n \mathrm{g} b_n, z_n - b_n\rangle.\end{aligned}$$

Using condition (iii) of Lemma (2.1) again, this yields

$$\begin{aligned}\langle c_n - b_n - \lambda_n \mathrm{g}(b_n), z_n - b_n\rangle &= \langle c_n - \lambda_n \mathrm{g}(c_n) - b_n, z_n - b_n\rangle + \langle \lambda_n \mathrm{g}(c_n) - \lambda_n \mathrm{g}(b_n), z_n - b_n\rangle \\ &\leq \langle \lambda_n \mathrm{g}(c_n) - \lambda_n \mathrm{g}(b_n), z_n - b_n\rangle \\ &\leq \lambda_n k\|c_n - b_n\| \cdot \|z_n - b_n\|\end{aligned}$$

and so,

$$\|z_n - a\|^2 \leq \|c_n - a\|^2 - \|c_n - b_n\|^2 - \|b_n - z_n\|^2 + 2\lambda_n k\|c_n - b_n\| \cdot \|z_n - b_n\|$$

for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}0 &\leq (\|z_n - b_n\| - \lambda_n k\|c_n - b_n\|)^2 \\ &= \|z_n - b_n\|^2 - 2\lambda_n k\|z_n - b_n\|.\|c_n - b_n\| + \lambda_n^2 k^2\|c_n - b_n\|^2,\end{aligned}$$

that is,

$$2\lambda_n k\|z_n - b_n\| \cdot \|c_n - b_n\| \leq \|z_n - b_n\|^2 + \lambda_n^2 k^2\|c_n - b_n\|^2,$$

implies

$$\begin{aligned}\|z_n - a\|^2 &\leq \|c_n - a\|^2 - \|c_n - b_n\|^2 - \|b_n - z_n\|^2 \\ &\quad + \lambda_n^2 k^2\|c_n - b_n\|^2 + \|z_n - b_n\|^2 \\ &\leq \|c_n - a\|^2 + \left(\lambda_n^2 k^2 - 1\right)\|c_n - b_n\|^2 \\ &\leq \|c_n - a\|^2.\end{aligned} \tag{3.10}$$

Now, by Eqs (3.6) and (3.10), we have

$$\begin{aligned}\|a_{n+1} - a\|^2 &= \|(\alpha_n c_n + (1-\alpha_n)\mathrm{S} z_n) - a\|^2 \\ &= \|\alpha_n(c_n - a) + (1-\alpha_n)(\mathrm{S} z_n - a)\|^2 \\ &\leq \alpha_n\|c_n - a\|^2 + (1-\alpha_n)\|\mathrm{S} z_n - a\|^2 - \alpha_n(1-\alpha_n)\|c_n - a - (\mathrm{S} z_n - a)\|^2 \\ &\leq \alpha_n\|c_n - a\|^2 + (1-\alpha_n)\|\mathrm{S} z_n - \mathrm{S} a\|^2 \\ &\leq \alpha_n\|c_n - a\|^2 + (1-\alpha_n)\|z_n - a\|^2 \\ &\leq \alpha_n\|c_n - a\|^2 + (1-\alpha_n)[\|c_n - a\|^2 + (\lambda_n^2 k^2 - 1)\|c_n - b_n\|^2]\end{aligned}$$

$$
\begin{aligned}
&\leq \|c_n - a\|^2 + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|c_n - b_n\|^2 \\
&\leq \|c_n - a\|^2 \\
&\leq \|a_n - a\|^2. \qquad (3.11)
\end{aligned}
$$

Hence, there exists a constant $s \geq 0$ such that

$$
\lim_{n\to\infty} \|a_n - a\| = s, \qquad (3.12)
$$

implies $\{a_n\}$ is bounded. This gives us that, $\{b_n\}, \{c_n\}, \{u_n\}$ and $\{v_n\}$ are all bounded.
From Eqs (3.7) and (3.11), we deduce

$$
\begin{aligned}
\beta_n(1-\beta_n)\|u_n - V_n u_n\|^2 &\leq \|a_n - a\|^2 - \|c_n - a\|^2 \\
&\leq \|a_n - a\|^2 - \|a_{n+1} - a\|^2.
\end{aligned}
$$

By using Eq (3.12), we find

$$
(u_n - V_n u_n) \to 0 \text{ as } n \to \infty.
$$

From Eq (3.3), we calculate

$$
u_n - c_n = \beta_n(u_n - V_n u_n) \to 0 \text{ as } n \to \infty. \qquad (3.13)
$$

Using Eqs (3.5), (3.6) in (3.11), we get

$$
\begin{aligned}
\|a_{n+1} - a\|^2 &\leq \|c_n - a\|^2 \\
&\leq \|a_n - a\|^2 - \|u_n - v_n\|^2 \\
\implies \|u_n - v_n\|^2 &\leq \|a_n - a\|^2 - \|a_{n+1} - a\|^2 \\
&\leq \|a_n - a_{n+1}\|\{\|a_n - a\| + \|a_{n+1} - a\|\} \\
&\to 0 \text{ as } n \to \infty \\
\implies u_n - v_n &\to 0 \text{ as } n \to \infty. \qquad (3.14)
\end{aligned}
$$

Again, using Eqs (3.4), (3.5), (3.6) in (3.11), we obtain

$$
\begin{aligned}
\|a_{n+1} - a\|^2 &\leq \|a_n - a\|^2 \leq \|u_n - a\|^2 \\
&\leq \|v_n - a\|^2 \\
&\leq \|a_n - a\|^2 - \|a_n - v_n\|^2 \\
\implies \|a_n - v_n\|^2 &\leq \|a_n - a\|^2 - \|a_{n+1} - a\|^2 \\
\implies a_n - v_n &\to 0 \text{ as } n \to \infty. \qquad (3.15)
\end{aligned}
$$

By triangle inequality, we find

$$
\begin{aligned}
\|a_n - u_n\| &\leq \|a_n - v_n\| + \|v_n - u_n\| \\
&\to 0 \text{ as } n \to \infty. \qquad (3.16)
\end{aligned}
$$

Also,

$$
\begin{aligned}
\|a_n - c_n\| &\leq \|a_n - v_n\| + \|v_n - u_n\| + \|u_n - c_n\| \\
&\to 0 \text{ as } n \to \infty. \qquad (3.17)
\end{aligned}
$$

Equations (3.6), (3.11) and (3.17), implies

$$
(1-\alpha_n)(1-\lambda_n^2 k^2)\|c_n - b_n\|^2 \leq \|c_n - a\|^2 - \|a_{n+1} - a\|^2
$$

and so

$$
c_n - b_n \to 0 \text{ as } n \to \infty.
$$

Thus,

$$\begin{aligned}\|a_n - b_n\| &\leq \|a_n - v_n\| + \|v_n - u_n\| + \|u_n - c_n\| + \|c_n - b_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.18}$$

Also, by definition of $\{b_n\}$, we have

$$\begin{aligned}\|b_n - z_n\|^2 &= \|P_C(c_n - \lambda_n g(c_n)) - P_C(c_n - \lambda_n g(b_n))\|^2 \\ &\leq \|(c_n - \lambda_n g(c_n)) - (c_n - \lambda_n g(b_n))\|^2 \\ &= \|\lambda_n g(c_n) - \lambda_n g(b_n)\|^2 \\ &\leq \lambda_n^2 k^2 \|c_n - b_n\|^2,\end{aligned}$$

implies, $b_n - z_n \to 0$ as $n \to \infty$. Again using triangle inequality, we have

$$\|c_n - z_n\| \leq \|c_n - b_n\| + \|b_n - z_n\|$$

and

$$\|v_n - z_n\| \leq \|v_n - u_n\| + \|u_n - c_n\| + \|c_n - z_n\|$$

gives, when $n \to \infty$

$$\|c_n - z_n\| \text{ and } \|v_n - z_n\| \to 0. \tag{3.19}$$

From the definition of $\{c_n\}$, we implies

$$(1 - \mu_n)(u_n - P_C(u_n - \gamma_n B^*(I - \mathrm{T})Bu_n)) = c_n - u_n.$$

Thus, Equation (3.13) gives

$$u_n - P_C(u_n - \gamma_n B^*(I - \mathrm{T})Bu_n) \to 0 \text{ as } n \to \infty. \tag{3.20}$$

Let $z \in \omega_w(u_n)$. Then there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ which weakly convergent to z. We acquire that $\{B^*(I - \mathrm{T})Bu_n\}$ is bounded, reason being $B^*(I - \mathrm{T})B$ is $\frac{1}{2\|B\|^2}$ inverse strongly monotone. From the firm nonexpansive nature of P_C, we have

$$\begin{aligned}&\|P_C(I - \gamma_{n_i} B^*(I - \mathrm{T})B)u_{n_i} - P_C(I - \hat{\gamma} B^*(I - \mathrm{T})B)u_{n_i}\| \\ &\qquad\qquad \leq |\gamma_{n_i} - \hat{\gamma}| \cdot \|B^*(I - \mathrm{T})Ba_{n_i}\|.\end{aligned}$$

Without loss of generality, we assume that $\gamma_{n_i} \to \hat{\gamma} \in \left(0, \frac{1}{\|B\|^2}\right)$ and so,

$$P_C(I - \gamma_{n_i} B^*(I - \mathrm{T})B)u_{n_i} - P_C(I - \hat{\gamma} B^*(I - \mathrm{T})B)u_{n_i} \to 0 \text{ as } i \to \infty. \tag{3.21}$$

Consider

$$\begin{aligned}&\|a_{n_i} - P_C(I - \hat{\gamma} B^*(I - \mathrm{T})B)a_{n_i}\| \\ &\quad \leq \|(a_{n_i} - u_{n_i}) + (u_{n_i} - P_C(I - \hat{\gamma} B^*(I - \mathrm{T})B)u_{n_i}) \\ &\qquad + (P_C(I - \hat{\gamma} B^*(I - \mathrm{T})B)u_{n_i} - P_C(I - \hat{\gamma} B^*(I - \mathrm{T})B)a_{n_i})\| \\ &\quad \leq \|a_{n_i} - u_{n_i}\| + \|u_{n_i} - P_C(I - \hat{\gamma} B^*(I - \mathrm{T})B)u_{n_i}\| + \|u_{n_i} - a_{n_i}\|.\end{aligned} \tag{3.22}$$

In particular

$$\|u_{n_i} - P_C(I - \hat{\gamma} B^*(I - \mathrm{T})B)u_{n_i}\|$$

$$\leq \|u_{n_i} - P_C(I - \gamma_{n_i} B^*(I - \mathrm{T})B)u_{n_i}\|$$
$$+ \|P_C(I - \gamma_{n_i} B^*(I - \mathrm{T})B)u_{n_i} - P_C(I - \hat{\gamma} B^*(I - \mathrm{T})B)u_{n_i}\|. \tag{3.23}$$

From Eqs (3.18), (3.20) and (3.21) in (3.23), we obtain

$$u_{n_i} - P_C(I - \hat{\gamma} B^*(I - \mathrm{T})B)u_{n_i} \to 0 \text{ as } i \to \infty. \tag{3.24}$$

Now, using Eq (3.16), (3.24) in (3.22), we find

$$a_{n_i} - P_C(I - \hat{\gamma} B^*(I - \mathrm{T})B)a_{n_i} \to 0 \text{ as } i \to \infty. \tag{3.25}$$

By the demiclosedness principle [8], (3.24) and (3.25), respectively, implies

$$z \in Fix(P_C(I - \hat{\gamma} B^*(I - \mathrm{T})B)),$$
$$z \in Fix(P_C(I - \hat{\gamma} B^*(I - \mathrm{T}_r^F)B)).$$

From Corollary 2.9 [11] and Lemma (2.4), we obtain

$$z \in C \cap B^{-1}(Fix(\mathrm{T})),$$
$$z \in C \cap B^{-1}(Fix(\mathrm{T}_r^F)) = z \in C \cap B^{-1}(EP(F))).$$

Now we claim that $z \in VI(C, \mathrm{g})$. From Eqs (3.13), (3.14), (3.17), (3.18) and (3.19), we obtain $b_{n_i} \to u$, $c_{n_i} \to u, z_{n_i} \to u, u_{n_i} \to u$ and $v_{n_i} \to u$. Interpret the set-valued mapping $A : \mathbb{H} \to \mathbb{H}$ by

$$Av = \begin{cases} \mathrm{g}(v) + N_C v, & if\ \forall\ v \in C \\ \phi, & if\ \forall\ v \notin C \end{cases}$$

In 2006, Takahashi [14] suggested that A is maximal monotone and for this $0 \in Av$ iff $v \in VI(C, \mathrm{g})$. For $(v, w) \in D(A)$ we have $w \in Av = \mathrm{g}(v) + N_C v$ and implies $w - \mathrm{g}(v) \in N_C v$. Therefore, for any $x \in C$, we get

$$\langle v - x, w - \mathrm{g}(v) \rangle \geq 0. \tag{3.26}$$

As $v \in C$. The explanation of b_n and Lemma (2.1) implies that

$$\langle c_n - \lambda_n \mathrm{g}(c_n) - b_n, b_n - v \rangle \geq 0.$$

Continuing

$$\left\langle \frac{c_n - b_n}{\lambda_n} + \mathrm{g}(c_n), v - b_n \right\rangle \geq 0.$$

By Equation (3.25) with $\{b_{n_i}\}$, we obtain

$$\langle w - \mathrm{g}(v), v - b_{n_i} \rangle \geq 0.$$

Thus,

$$\langle w, v - b_{n_i} \rangle \geq \langle \mathrm{g}(v), v - b_{n_i} \rangle$$
$$\geq \langle \mathrm{g}(v), v - b_{n_i} \rangle - \left\langle \frac{c_{n_i} - b_{n_i}}{\lambda_{n_i}} + \mathrm{g}(c_{n_i}), v - b_{n_i} \right\rangle$$

$$= \langle g(v) - g(b_{n_i}), v - b_{n_i}\rangle + \langle g(b_{n_i}) - g(c_{n_i}), v - b_{n_i}\rangle - \left\langle \frac{c_{n_i} - b_{n_i}}{\lambda_{n_i}}, v - b_{n_i}\right\rangle$$
$$\geq \langle g(b_{n_i}) - g(c_{n_i}), v - b_{n_i}\rangle - \left\langle \frac{c_{n_i} - b_{n_i}}{\lambda_{n_i}}, v - b_{n_i}\right\rangle.$$

By considering $i \to \infty$, we have

$$\langle w, v - z\rangle \geq 0.$$

By maximal monotonicity of A, we obtain $0 \in Az$ and then $z \in VI(C, g)$.
Now, we will exhibit that $z \in Fix(S)$.
From Eqs (3.6), (3.10) and the nonexpansive nature of S, we get

$$\|S(z_n) - a\| = \|S(z_n) - S(a)\| \leq \|z_n - a\| \leq \|c_n - a\| \leq \|a_n - a\|.$$

By taking limit superior

$$\lim_{n\to\infty} \|S(z_n) - a\| \leq c$$

and

$$\lim_{n\to\infty} \|c_n - a\| \leq c.$$

Further

$$\begin{aligned} \lim_{n\to\infty} \|\alpha_n(c_n - a) + (1 - \alpha_n)(S(z_n) - a)\| &= \lim_{n\to\infty} \|\alpha_n c_n + (1 - \alpha_n)Sz_n - a\| \\ &= \lim_{n\to\infty} \|a_{n+1} - a\| \\ &= c. \end{aligned}$$

Thus, Lemma (2.6), implies

$$\lim_{n\to\infty} \|Sz_n - c_n\| = 0. \tag{3.27}$$

Again from the fact of

$$\begin{aligned} \|S(c_n) - c_n\| &= \|(S(c_n) - S(z_n)) + (S(z_n) - c_n)\| \\ &\leq \|S(c_n) - S(z_n)\| + \|S(z_n) - c_n\| \\ &\leq \|c_n - z_n\| + \|S(z_n) - c_n\|. \end{aligned}$$

By Eqs (3.19) and (3.27), we find

$$\lim_{n\to\infty} \|S(c_n) - c_n\| = 0.$$

This infers that

$$\lim_{i\to\infty} \|(I - S)c_{n_i}\| = \lim_{i\to\infty} \|c_{n_i} - Sc_{n_i}\| = 0.$$

Thus, we have $z \in Fix(S)$.
Now, we prove $z \in EP(f)$. As $u_n = T_r^f a_n$ and

$$f(u_n, z) + \frac{1}{r}\langle z - u_n, u_n - a_n\rangle \geq 0 \;\forall\; z \in C.$$

From monotonicity of f, we have

$$\frac{1}{r}\langle z - u_n, u_n - a_n\rangle \geq -f(u_n, z) \geq f(z, u_n)$$

and hence

$$\left\langle z - u_{n_i}, \frac{u_{n_i} - a_{n_i}}{r}\right\rangle \geq f(z, u_{n_i}).$$

Since $\frac{u_{n_i}-a_{n_i}}{r} \to 0$ as $u_{n_i} \to 0$ weakly lower semicontinuity of $f(a, y)$ on second variable y, we have

$$f(z, u) \leq 0 \ \forall \ z \in C.$$

For k with $0 \leq k \leq 1$ and $z \in C$, let $z_t = kz + (1-t)u$

$$\begin{aligned} &0 = f(z_t, z_t) \leq tf(z_t, z) + (1-t)f(z_t, u) \leq tf(z_t, z) \\ \Longrightarrow\ & f(z_t, z) \geq 0 \\ \Longrightarrow\ & f(u, z) \geq 0 \\ & u \in EP(f). \end{aligned}$$

Consequently, $w_w(a_n) \subset \daleth$. From the Lemma (2.7), the sequence $\{a_n\}$ converges weakly to an element $u \in \daleth$ and Lemma 3.2 [13] satisfies $u = \lim_{n\to\infty} P_\daleth a_n$. □

The successive result gives us the suitable conditions to obtain the presence of a common solution of the split variational inequality problems, fixed point problems and split equilibrium problems , that is, to discover an element a^* in such a way that

$$a^* \in VI(C, \mathrm{g}) \cap Fix(\mathrm{S}) \cap EP(f) \text{ and } Ba^* \in VI(Q, \mathrm{g}') \cap EP(F).$$

Theorem 3.2. *Set* $\daleth = \{u \in VI(C, \mathrm{g}) \cap Fix(\mathrm{S}) \cap EP(f) : Bu \in VI(Q, \mathrm{g}') \cap EP(F)\}$ *and consider that* $\daleth \neq \phi$. *Let the sequences* $\{u_n\}, \{v_n\}, \{a_n\}, \{b_n\}$ *and* $\{c_n\}$ *be defined by* $a_1 = a \in C$ *and*

$$\begin{aligned} & f(v_n, z) + \frac{1}{r_n}\langle z - v_n, v_n - a_n\rangle \geq 0, \\ & F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - Bv_n\rangle \geq 0, \\ & c_n = \mu_n u_n + (1-\mu_n)P_C(u_n - \gamma_n B^*(I - P_Q(I - \theta\mathrm{g}'))Bu_n), \\ & b_n = P_C(c_n - \lambda_n \mathrm{g}(c_n)), \\ & a_{n+1} = \alpha_n c_n + (1-\alpha_n)\mathrm{S}P_C(c_n - \lambda_n \mathrm{g}(b_n)), \end{aligned}$$

for all $n \in \mathbb{N}$. *Then, the sequence* $\{a_n\}$ *weakly converges to an element* $u \in \daleth$, *where* $u = \lim_{n\to\infty} P_\daleth a_n$.

Proof. From the δ-inverse strongly monotonicity of g', it is $\frac{1}{\delta}$- Lipschitz continuous and so, $\theta \in (0, 2\delta)$, we find that $I - \theta\mathrm{g}'$ is nonexpansive . Also, P_Q is firmly nonexpansive, implies $P_Q(I - \theta\mathrm{g}')$ is nonexpansive . With replacement of $\mathrm{T} = P_Q(I - \theta\mathrm{g}')$ in Theorem (3.1), we get that $\{a_n\}$ is weakly convergent to an element $u \in VI(C, \mathrm{g}) \cap Fix(\mathrm{S}) \cap EP(f)$ and $Bu \in Fix(P_Q(I - \theta\mathrm{g}') \cap \mathrm{T}^F_{r_n})$. We pursue from $Bu = P_Q(I - \theta\mathrm{g}')Bu$ and $Bu \in EP(F)$ and Lemma (2.1) that $Bu \in VI(Q, \mathrm{g}') \cap EP(F)$. This completes the proof. □

The following results are the direct consequences of Theorem (3.1).

Theorem 3.3. *Let* $A : \mathbb{H}_2 \to \mathbb{H}_2$ *be a maximal monotone mapping with* $D(A) \neq \phi$. *Consider* $\daleth = \{u \in VI(C, \mathrm{g}) \cap Fix(\mathrm{S}) \cap EP(f) : Bu \in A^{-1}0 \cap EP(F)\}$ *and assume* $\daleth \neq \phi$. *Let the sequences* $\{u_n\}, \{v_n\}, \{a_n\}, \{b_n\}$ *and* $\{c_n\}$ *be defined by* $a_1 = a \in C$ *and*

$$\begin{aligned} & f(v_n, z) + \frac{1}{r_n}\langle z - v_n, v_n - a_n\rangle \geq 0, \\ & F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - Bv_n\rangle \geq 0, \end{aligned}$$

$$
\begin{aligned}
&c_n = \mu_n u_n + (1-\mu_n) P_C(u_n - \gamma_n B^*(I - J_r) B u_n),\\
&b_n = P_C(c_n - \lambda_n \mathrm{g}(c_n)),\\
&a_{n+1} = \alpha_n c_n + (1-\alpha_n)\mathrm{S} P_C(c_n - \lambda_n \mathrm{g}(b_n)),
\end{aligned}
$$

for all $n \in \mathbb{N}$, where $J - r$ is resolvent of A for $r > 0$. Then, the sequence $\{a_n\}$ weakly converges to an element $u \in \daleth$, where $u = \lim_{n\to\infty} P_{\neg} a_n$.

Proof. From the firmly nonexpansive nature of J_r and $Fix(J_r) = A^{-1}0$, the proof remains the same as of Theorem (3.1) by considering $J_r = \mathrm{T}$. □

Theorem 3.4. *Let $A : \mathbb{H}_2 \to \mathbb{H}_2$ be a maximal monotone mapping with $D(A) \neq \phi$ and $\mathrm{G} : \mathbb{H}_2 \to \mathbb{H}_2$ be a δ-inverse strongly monotone mapping . Set $\daleth = \{u \in VI(C, \mathrm{g}) \cap Fix(\mathrm{S}) \cap EP(f) : Bu \in (A + \mathrm{G})^{-1}0 \cap EP(F)\}$ and assume $\daleth \neq \phi$. Let the sequences $\{u_n\}, \{v_n\}, \{a_n\}, \{b_n\}$ and $\{c_n\}$ be defined by $a_1 = a \in C$ and*

$$
\begin{aligned}
&f(v_n, z) + \frac{1}{r_n}\langle z - v_n, v_n - a_n\rangle \geq 0,\\
&F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - Bv_n\rangle \geq 0,\\
&c_n = \mu_n u_n + (1-\mu_n) P_C(u_n - \gamma_n B^*(I - J_r(I - r\mathrm{G})) B u_n),\\
&b_n = P_C(c_n - \lambda_n \mathrm{g}(c_n)),\\
&a_{n+1} = \alpha_n c_n + (1-\alpha_n)\mathrm{S} P_C(c_n - \lambda_n \mathrm{g}(b_n)),
\end{aligned}
$$

for all $n \in \mathbb{N}$, where $J - r$ is resolvent of A for $r > 0$. Then, the sequence $\{a_n\}$ weakly converges to an element $u \in \daleth$, with $u = \lim_{n\to\infty} P_{\neg} a_n$.

Proof. From the $\delta-$ strongly inverse monotone nature of G implies that $I - r\mathrm{G}$ is nonexpansive. Also, from the nonexpansive nature of J_r, we get that $J_r(I - r\mathrm{G})$ is also nonexpansive . As $u \in (A + \mathrm{G})^{-1}0$ if and only if $u = J_r(I - r\mathrm{G})u$. Thus, the proof remains the same as of Theorem (3.1) by considering $J_r(I - r\mathrm{G}) = \mathrm{T}$. □

4. Conclusions

We obtained the weak convergence of the defined algorithm for solving variational inequality, split common fixed point and split equilibrium problems, by extending the results of Tian et al. [11] and Lohewech et al. [12].

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Conflict of interest

There is no conflict of interest.

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