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Best Proximity Point for Generalized Rational α_s-Proximal Contraction

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Abstract

Best proximity point problem in S-M(S-metric) spaces is thought to be a generalization of a G- metric spaces. In this study, we provide proof a best proximity points theorem of α_S−Proximal mapping admissible and its several types by generalizing the theory of α−admissible mapping in S-M spaces. We present generalized rational α_S−Proximal contraction type mappings and investigate the best proximity point in S-M spaces. In addition, we provide an illustration to show how the result can beused.

MSC: 47H10;54H25
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1 Introduction

The best approximation results offer an approximation solution to fixed point equation $T\zeta = \zeta$, when a nonself-mapping T has no fixed point. A well-known best approximation theorem in particular, due to Fan [6], reveals the fact that " if K is a non-empty compact convex subset of a Hausdorff locally convex topological vector space X and T : K → X is a continous mapping, then there exists an element x satisfying the condition $d(\zeta, T\zeta) = \inf\{d(\mu, T\zeta) : \mu \in K\}$, where d is a metric on X ".

As a generalization of the idea of the best approximation, the best proximity point theory has evolved. The best proximity point theorem is taken into consideration when addressing a complication to discover an approximate solution that is optimal since it ensures the existence of an approximate solution.

Banach Contraction principle is important for finding a fixed point. Due to its diversity, simplicity, and ease of application, many scholars consider it to be one of the most fascinating topics.. In various

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ways, they tried to apply the Banach contraction principle. Samet et al. [18] introduced the concepts of α−admissible mapping and α-ψ-contractive mappings in metric spaces. Findings of Samet et al. [18] demonstrated that Banach's fixed point theorem and a number of other findings are immediate results of their findings. But on the other hand, Sedghi et al. [19] established the idea of S-M spaces as one outcome of the generalization of metric spaces.

Let B and C be two non-empty subsets of a metric space (X, d). Choose an element $\zeta \in B$ is referred to as a fixed point on a certain map. $T : B \to C$ if $T(b) = b$. Certainly, $T(B) \cap B \neq \phi$ is a necessary (but not sufficient) situation for the existence of a fixed point of T. If $T(B) \cap B = \phi$, then $d(\zeta, T\zeta) \geq 0$ for all $\zeta \in B$ that is, the set of fixed points of T is empty. Under such circumstances, one frequently tries to find an element ζ which is in some sense closest to $T\zeta$. Best proximity point analysis has been developed inthis direction.

Choose an element $b \in B$ is called a best proximity point of T if

$$d(b, TB) = d(B, C),$$

where

$$d(B, C) = \inf\{d(\zeta, \mu) : \zeta \in B, \mu \in C\}.$$

The reason being that $d(\zeta, T\zeta) \geq d(B, C)$ for all $\zeta \in B$, the global minimum of the mapping $\zeta \to d(\zeta, T\zeta)$ is attained at the Best proximity point.

Hussain et al. [9] proved certain Best proximity point results in the setting of G-metric spaces. Mo- tivated by inspiration by Hussain et al. [9] and Sedghi et al. [19], In this paper, we prove some best proximity point results in S-M spaces.

2 PRELIMINARIES

Initially, we must remember a few crucial definition's, lemma's and results for this the notion of S-M spaces as described below.

Definition 2.1. *[18] "Let* $T : X \to X$ *be a self-mapping on a metric space* (X, d)*, and let* $\alpha : X \times X \to [0, +\infty)$ *be a function. It is said that* T *is α-admissible if* $\zeta, \mu \in X$,

$$\alpha(\zeta, \mu) \geq 1 \implies \alpha(T\zeta, T\mu) \geq 1.\text{"}$$

Example 2.2. *"Consider* $X = [0, +\infty)$*, and define* $T : X \to X$ *and* $\alpha : X \times X \to [0, +\infty)$ *by* $T\zeta = 5\zeta$ *for all* $\zeta, \mu \in X$

$$\alpha(\zeta, \mu) = \begin{cases} e^{\frac{\zeta}{\mu}} & \text{if } \zeta \geq \mu \ \zeta \neq 0 \\ 0 & \text{if } \zeta < \mu \end{cases}$$

Then T *is α-admissible."*

Definition 2.3. *[17] "Let* T *be a self-mapping on a metric space* (X, d)*, and let* $\alpha, \eta : X \times X \to [0, +\infty)$ *be two functions.* T *is said to be an α-admissible mapping with respect to η if* $\zeta, \mu \in X$, $\alpha(\zeta, \mu) \geq \eta(\zeta, \mu)$

imply $\alpha(\mathrm{T}\zeta, \mathrm{T}\mu) \geq \eta(\mathrm{T}\zeta, \mathrm{T}\mu)$.

It can be noted that if we take $\eta(\zeta, \mu) = 1$, *then this definition reduces to Definition 2.1. Also, if wetake* $\alpha(\zeta, \mu) = 1$, *then* T *is said to be an* η*-subadmissible mapping."*

Definition 2.4. *[11] "Let* $\mathrm{T} : \mathrm{B} \to \mathrm{C}$, $\alpha : \mathrm{B} \times \mathrm{B} \to [0, +\infty)$. *We say that* T *is* α*-Proximal admissiblemapping if*

$$\left.\begin{array}{l}\alpha(\zeta_1, \zeta_2) \geq 1, \\ d(u_1, \mathrm{T}\zeta_1) = d(\mathrm{B}, \mathrm{C}), \\ d(u_2, \mathrm{T}\zeta_2) = d(\mathrm{B}, \mathrm{C}),\end{array}\right\} \Longrightarrow \alpha(u_1, u_2) \geq 1$$

for all $\zeta_1, \zeta_2, u_1, u_2 \in A$*."*

Certainly if B = C then α-Proximal admissible map T converted to α-admissible map.

Definition 2.5. *[8] "Let* $\mathrm{T} : \mathrm{B} \to \mathrm{C}$, *and* $\alpha, \eta : \mathrm{B} \times \mathrm{B} \to [0, +\infty)$ *be functions. We say that* T *is* α*-proximal admissible with respect to* η *if, for all* $\zeta_1, \zeta_2, u_1, u_2 \in \mathrm{B}$,

$$\left.\begin{array}{l}\alpha(\zeta_1, \zeta_2) \geq \eta(\zeta_1, \zeta_2), \\ d(u_1, T\zeta_1) = d(\mathrm{B}, \mathrm{C}), \\ d(u_2, T\zeta_2) = d(\mathrm{B}, \mathrm{C}),\end{array}\right\} \Longrightarrow \alpha(u_1, u_2) \geq \eta(u_1, u_2).$$

Note that if we take $\eta(\zeta, \mu) = 1$ *for all* $\zeta, \mu \in \mathrm{B}$, *then this definition reduces to Definition 2.4. In case* $\alpha(\zeta, \mu) = 1$ *for all* $\zeta, \mu \in \mathrm{B}$, *then we shall say that* T *is* η*-Proximal subadmissible mapping."*

$\mathrm{G} = \{g : [0, +\infty) \to [0, 1)$ that way $g(t_n) \to 1$ implies $t_n \to 0\}$

Definition 2.6. *[13] "A mapping* $\mathrm{T} : \mathrm{B} \to \mathrm{C}$, *is called Geraghty's proximal contraction of the first kind if,there exists* $\beta \in \mathrm{G}$ *such that*

$$\left.\begin{array}{l}d(u, Tx) = d(A, B) \\ d(v, Ty) = d(A, B)\end{array}\right\} \Rightarrow d(u, v) \leq \beta(d(x, y))d(x, y)$$

for all $u, v, x, y \in A$*."*

Definition 2.7. *[13] "A mapping* $\mathrm{T} : \mathrm{B} \to \mathrm{C}$, *is called Geraghty's proximal contraction of the second kindif, there exists* $\beta \in \mathrm{G}$ *such that*

$$\left.\begin{array}{l}d(u, \mathrm{T}\zeta) = d(\mathrm{B}, \mathrm{C}) \\ d(u, \mathrm{T}\mu) = d(\mathrm{B}, \mathrm{C})\end{array}\right\} \Rightarrow d(\mathrm{T}u, \mathrm{T}v) \leq \beta(d(\mathrm{T}\zeta, T\mu))d(\mathrm{T}\zeta, \mathrm{T}\mu)$$

for all $u, v, \zeta, \mu \in \mathrm{B}$*."*

Definition 2.8. *[19] " Let* X *be a non-empty set. An S-M on* X *is a function* $S : \mathrm{X} \times \mathrm{X} \times \mathrm{X} \to [0, +\infty)$ *that satisfies the following conditions for each* $\zeta, \mu, \varrho, b \in \mathrm{X}$:

1. $S(\zeta, \mu, \varrho) \geq 0$,
2. $S(\zeta, \mu, \varrho) = 0$ *if and only if* $\zeta = \mu = \varrho$,
3. $S(\zeta, \mu, \varrho) \leq S(\zeta, \zeta, b) + S(\mu, \mu, b) + S(\varrho, \varrho, b)$.

The pair (X, S) *is called S-M space."*

This assertion is an emphasis of G-metric spaces [14] and D^*-metric spaces [20]. Realize that each S-Mon X induces a metric d_s on X as explained by

$d_s(\zeta, \mu) = S(\zeta, \zeta, \mu) + S(\mu, \mu, \zeta)$, for all $\zeta, \mu \in X$.

Example 2.9. *[19] " Let* $X=\mathbb{R}$*. Then*

$$S(\zeta, \mu, \varrho) = |\zeta - \mu| + |\mu - \varrho|$$

for all $\zeta, \mu, \varrho \in \mathbb{R}$*, is an S-M on* X*."*

Example 2.10. *[19]" Let* $X = \mathbb{R}^2$ *and d is ordinary metric on* X*. Put*

$$S(\zeta, \mu, \varrho) = d(\zeta, \mu) + d(\zeta, \varrho) + d(\mu, \varrho)$$

for all $\zeta, \mu, \varrho \in \mathbb{R}$*. Then S is an S-M on* X*."*

Lemma 2.11. *[19] " Let* (X, S) *be an S-M space. Then* $S(\zeta, \zeta, \mu) = S(\mu, \mu, \zeta)$*, for all* $\zeta, \mu \in X$*."*

Lemma 2.12. *[7] " Let* (X, S) *be an S-M space. Then*

$$S(\zeta, \zeta, \varrho) \le 2S(\zeta, \zeta, \mu) + S(\mu, \mu, \varrho) \text{ and } S(\zeta, \zeta, \varrho) \le 2S(\zeta, \zeta, \mu) + S(\varrho, \varrho, \mu)$$

for all $\zeta, \mu, \varrho \in X$*."*

Definition 2.13. *[19] Let* (X, S) *be an S-M space.*

1. *" A sequence* $\{\zeta_l\}$ *in* X *converges to* ζ *if and only if* $S(\zeta_l, \zeta_l, \zeta) \to 0$ *as* $l \to +\infty$*. That is, for each* $\epsilon > 0$*, there exists* $l_0 \in \mathbb{N}$ *such that, for all* $l \ge l_0$, $S(\zeta_l, \zeta_l, \zeta) < \epsilon$*, and we denote this by* $\lim_{l\to+\infty} \zeta_l = \zeta$*."*
2. *" A sequence* $\{\zeta_l\}$ *in* X *is called a Cauchy sequence if for each* $\epsilon > 0$ *there exists* $l_0 \in \mathbb{N}$ *such that* $S(\zeta_l, \zeta_l, \zeta_m) < \epsilon$ *for each* $l, m \ge l_0$*."*
3. *" That S-M space* (X, S) *is said to be complete if every Cauchy sequence is convergent."*

We now consider the meaning of α_s*-admissible mappings and their generalizations in S-M spaces.In this article, we present a number of concepts of* α*-admissible mappings in the context of S-M spaces and name them* α_s*-admissible.*

Definition 2.14. *[21] " Let* $T : X \to X$ *and* $\alpha : X^3 \to [0, +\infty)$*. Then* T *is said to be* α*–admissible if forall* $\zeta, \mu, \varrho \in X$

$\alpha(\zeta, \mu, \varrho) \ge 1$ *implies* $\alpha(T\zeta, T\mu, T\varrho) \ge 1$*."*

Definition 2.15. *[21] " Let* (X, S) *be an S-M space,* $T : X \to X$ *and* $\alpha_s : X \times X \times X \to [0, +\infty)$*. Then* T *is called* α_s*– admissible if* $u, v, w \in X$,

$\alpha_s(u, v, w) \ge 1$ *implies* $\alpha_s(Tu, Tv, Tw) \ge 1$*."*

Example 2.16. *[16] " Consider* $X = [0, +\infty)$*. Define* $T : X \to X$ *and* $\alpha_s : X \times X \times X \to [0, +\infty)$ *by* $Tu = 4u$ *for all* $u, v, w \in X$ *and*

$$\alpha_s(u, v, w) = \begin{cases} e^{\frac{u}{v}} & \text{if } u \geq v \geq w \;\; v \neq 0 \\ 0 & \text{if } u < v < w \end{cases}$$

Then T *is* α_s*−admissible."*

Definition 2.17. *[16] "Let* (X, S) *be an S-metric space,* $T : X \to X$*, and let* $\alpha_s, \eta_s : X \times X \times X \to [0, +\infty)$ *be two functions. We say that* T *is an* α_s*−admissible mapping with respect to* η_s *if* $u, v, w \in X$*,* $\alpha_s(u, v, w) \geq \eta_s(u, v, w)$ *implies* $\alpha_s(Tu, Tv, Tw) \geq \eta_s(Tu, Tv, Tw)$*.*
Note that if we take $\eta_s(u, v, w) = 1$*, then this definition reduces to Definition 2.15. "*

Definition 2.18. *[15] "Let* (X , S) *be an S-M space and let* B *and* C *be two non-empty subsets of* X *. Then* C *is said to be approximatively compact with respect to* B *if every sequence* $\{\mu_l\}$ *in* C*, satisfying the condition* $d_s(\zeta, \mu_n) \to d_s(\zeta, C)$ *for some* ζ *in* B *has a convergent subsequence."*

3 Main Result

At first, we presume

$\Xi = \{\xi : [0, \infty) \to [0, \infty)$ such that ξ is non-decreasing and continous $\}$ where $\xi(x) = 0$ if and only if $x = 0$.

Definition 3.1. *Let* (X, S) *be a S-M space and let* B *and* C *be two non-empty subset of* X *then* $T : B \to C$ *and* $\alpha_s : B \times B \times B \to [0, +\infty)$*. We say* T *is* α_s *-Proximal admissible if*

$$\left.\begin{array}{l} \alpha_s(\zeta, \mu, \varrho) \geq 1, \\ d_s(\vartheta, T\zeta) = d_s(B, C), \\ d_s(\nu, T\mu) = d_s(B, C), \\ d_s(\kappa, T\varrho) = d_s(B, C), \end{array}\right\} \Longrightarrow \alpha_s(\vartheta, \nu, \kappa) \geq 1, \tag{3.1}$$

for all $\zeta, \mu, \varrho, \vartheta, \nu, \kappa \in B$*.*

Example 3.2. *Consider* $X = R$ *and let* a *be any fixed positive real number,* $B = \{(a, \mu, \varrho) : \mu, \varrho \geq 0\}$ *and* $C = \{(0, \mu, \varrho) : \mu, \varrho \geq 0\}$*. Define* $T : B \to C$ *by*

$$T(a, \mu, \varrho) = \begin{cases} (0, \frac{\mu}{4}, \varrho) & \text{if } \mu, \varrho \geq 0, \\ (0, 4\mu, \varrho) & \text{if } \mu, \varrho < 0. \end{cases}$$

Define $\alpha_s : B \times B \times B \to [0, +\infty)$ *by*

$$\alpha_s((a, \mu_1, \varrho_1), (a, \mu_2, \varrho_2), (a, \mu_3, \varrho_3)) = \begin{cases} 2 & \text{if } \mu_i, \varrho_i \geq 0 \text{ where } i = 1, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Then $S(\zeta, \mu, \varrho) = \frac{1}{4}(|\zeta - \varrho| + |\mu - \varrho|)$ *is S-M on* X*, let* $d_s(B, C) = |\zeta - \mu|$ *and* $\kappa_1 = (a, \mu_1, \varrho_1)$*,* $\kappa_2 = (a, \mu_2, \varrho_2)$*,* $\kappa_3 = (a, \mu_3, \varrho_3)$*,* $\kappa_4 = (a, \mu_4, \varrho_4)$*,* $\kappa_5 = (a, \mu_5, \varrho_5)$*,* $\kappa_6 = (a, \mu_6, \varrho_6)$ *be arbitrary points from* B *satisfying,*

$$\alpha_s(\kappa_1,\kappa_2,\kappa_3) = 2,$$
$$\text{so } \mu_1,\mu_2,\mu_3,\varrho_1,\varrho_2,\varrho_3 \geq 0,$$
$$d_s(\kappa_4, T\kappa_1) = a = d_s(B,C),$$
$$d_s(\kappa_5, T\kappa_2) = a = d_s(B,C),$$
$$d_s(\kappa_6, T\kappa_3) = a = d_s(B,C).$$

So further we solve $\mu_4 = \frac{\mu_1}{4}, \varrho_4 = \varrho_1, \mu_5 = \frac{\mu_2}{4}, \varrho_5 = \varrho_2$ *and* $\mu_6 = \frac{\mu_3}{4}, \varrho_6 = \varrho_3$ *which implies* $\mu_i, \varrho_i \geq 0$, *where* $i = 1,2,3$. *Hence* $\alpha_s(\kappa_4,\kappa_5,\kappa_6) = 2$. *Therefore,* T *is* α_s*-Proximal admissible map.*

Definition 3.3. *Choose* B *and* C *be two non-empty subsets of an S-M space* (X, S). *A non-self mapping* $T : B \to C$ *is called generalized rational* α_s*–Proximal contraction mapping if* $\alpha_s : B \times B \times B \to [0,+\infty)$ *is a function and there exist* $g \in G$ *and* $\xi \in X_i$ *such that, for all* $\zeta, \vartheta, \vartheta^*, \mu, \nu \in B$,

$$\left.\begin{array}{l} d_s(\vartheta, T\zeta) = d_s(B,C), \\ d_s(\vartheta^*, T\vartheta) = d_s(B,C), \\ d_s(\nu, T\mu) = d_s(B,C), \end{array}\right\} \Longrightarrow \alpha_s(\vartheta,\vartheta^*,\nu)\xi(S(\vartheta,\vartheta^*,\nu)) \leq g(\xi(\Delta(\zeta,\vartheta,\mu)))\xi(\Delta(\zeta,\vartheta,\mu)), \tag{3.2}$$

where

$$\Delta(\zeta,\vartheta,\mu) = \max\Big\{ S(\zeta,\zeta,\vartheta), S(\vartheta,\vartheta,\mu), S(\mu,\mu,\zeta), \frac{S(\zeta,\zeta,\vartheta)S(\vartheta,\vartheta,\mu)}{1+S(\zeta,\zeta,\vartheta)S(\vartheta,\vartheta,\mu)}, \frac{S(\vartheta,\vartheta,\mu)S(\mu,\mu,\zeta)}{1+S(\vartheta,\vartheta,\mu)S(\mu,\mu,\zeta)}, \frac{S(\mu,\mu,\zeta)S(\zeta,\zeta,\vartheta)}{1+S(\mu,\mu,\zeta)S(\zeta,\zeta,\vartheta)} \Big\} \tag{3.3}$$

Definition 3.4. *Let* (X, S) *be an S-M space,* $T : B \to C$, *and* $\alpha_s, \eta_s : B \times B \times B \to [0,+\infty)$. *We say* T *is* α_s *-Proximal admissible with respect to* η_s *if for all* $\zeta, \mu, \varrho, \vartheta, \nu, \kappa \in B$, *we have*

$$\left.\begin{array}{l} \alpha_s(\zeta,\mu,\varrho) \geq \eta_s(\zeta,\mu,\varrho), \\ d_s(\vartheta, T\zeta) = d_s(B,C), \\ d_s(\nu, T\mu) = d_s(B,C), \\ d_s(\kappa, T\varrho) = d_s(B,C), \end{array}\right\} \Longrightarrow \alpha_s(\vartheta,\nu,\kappa) \geq \eta_s(\vartheta,\nu,\kappa). \tag{3.4}$$

Recall that if we take $\eta_s(\vartheta,\nu,\kappa) = 1$, *then this definition converted to Definition 3.2. Also, if we take* $\alpha_s(\vartheta,\nu,\kappa) = 1$, *then we say that* T *is an* η_s*– Proximal subadmissible mapping.*

Theorem 3.5. *Let* B *and* C *be two non-empty subsets of an S-M space* (X, S) *such that* (B, S) *be a complete S-M space and* B_0 *be non-empty set.* B *and* C *are approximatively compact with respect to* B. *Let* $\alpha_s : B \times B \times B \to [0,+\infty)$ *be a function and* $T : B \to C$ *be a mapping then the following conditions hold:*

1. T *is a generalized rational* α_s*–Proximal contraction mapping.*
2. *There exists* $\zeta_0 \in B$ *such that* $\alpha_s(\zeta_0, \zeta_1, T\zeta_1) \geq 1$.
3. T *is continuous.*
4. *If* $\{\zeta_l\}$ *is a sequence in* B *such that* $\alpha_s(\zeta_l, \zeta_{l+1}, \zeta_{l+1}) \geq 1$ *for all* $l \in N \cup \{0\}$ *and* $\zeta_l \to \varrho \in B$ *as* $l \to +\infty$, *then there exists a subsequence* $\{\zeta_{m_l}\}$ *of* $\{\zeta_n\}$ *such that* $\alpha_s(\zeta_{m_l}, \varrho, \varrho) \geq 1$ *for all k.*

Suppose that $T(B_0) \subseteq C_0$. *Then* T *has the unique best proximity point that is,* $\varrho \in B$ *such that* $d_s(\varrho, T\varrho) = d_s(B,C)$.

Proof. Due to the subset B_0 is not empty, we choose ζ_0 in B_0. Taking $T\zeta_0 \in T(B_0) \subseteq C_0$ into account, wecan find $\zeta_1 \in B_0$ like that

$$d_s(\zeta_1, T\zeta_0) = d_s(B, C).$$

Moreover, given $T\zeta_1 \in T(B_0) \subseteq C_0$, Hence, there are elements ζ_2 and ζ_3 in B_0 such that

$$d_s(\zeta_2, T\zeta_1) = d_s(B, C),$$
$$d_s(\zeta_3, T\zeta_2) = d_s(B, C).$$

Repeating this process, we get a sequence $\{\zeta_l\}$ in B_0 satisfying

$$d_s(\zeta_{l+1}, T\zeta_l) = d_s(B, C), \forall l \in N \cup \{0\}.$$

By by taking $\vartheta = \zeta_l$, $\zeta = \zeta_{l-1}$, $\nu = \zeta_{l+1}$, $\mu = \zeta_l$,$\vartheta^* = \zeta_{l+1}$,
Equation 3.2 gives

$$\alpha_s(\zeta_l, \zeta_{l+1}, \zeta_{l+1})\xi(S(\zeta_l, \zeta_{l+1}, \zeta_{l+1})) \leq g(\xi(\Delta(\zeta_{l-1}, \zeta_l, \zeta_l)))(\xi(\Delta(\zeta_{l-1}, \zeta_l, \zeta_l)). \quad (3.5)$$

By the assumption $\alpha_s(\zeta_0, \zeta_1, \zeta_1) \geq 1$ and T is α_s−Proximal admissible, we have

$$\alpha_s(\zeta_l, \zeta_{l+1}, \zeta_{l+1}) \geq 1 \text{ for all } l \in N \cup \{0\},$$

$$\text{and } \xi(S(\zeta_l, \zeta_{l+1}, \zeta_{l+1})) \leq g(\xi(\Delta(\zeta_{l-1}, \zeta_l, \zeta_l)))\xi(\Delta(\zeta_{l-1}, \zeta_l, \zeta_l)). \quad (3.6)$$

where

$$\Delta(\zeta_{l-1}, \zeta_l, \zeta_l) = \max \Big\{ S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l), S(\zeta_l, \zeta_l, \zeta_l), S(\zeta_l, \zeta_l, \zeta_{l-1}), \frac{S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l)S(\zeta_l, \zeta_l, \zeta_l)}{1 + S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l)S(\zeta_l, \zeta_l, \zeta_l)}, \frac{S(\zeta_l, \zeta_l, \zeta_l)S(\zeta_l, \zeta_l, \zeta_{l-1})}{1 + S(\zeta_l, \zeta_l, \zeta_l)S(\zeta_l, \zeta_l, \zeta_{l-1})}, \frac{S(\zeta_l, \zeta_l, \zeta_{l-1})S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l)}{1 + S(\zeta_l, \zeta_l, \zeta_{l-1})S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l)} \Big\}$$
$$= \max\{S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l), S(\zeta_l, \zeta_l, \zeta_{l-1})\}.$$

If max $\{S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l), S(\zeta_l, \zeta_l, \zeta_{l-1})\} = S(\zeta_l, \zeta_l, \zeta_{l-1})$ then the Equation 3.6 becomes

$$\xi(S(\zeta_l, \zeta_{l+1}, \zeta_{l+1})) \leq g(\xi(S(\zeta_l, \zeta_l, \zeta_{l-1})))\xi(S(\zeta_l, \zeta_l, \zeta_{l-1}))$$
$$< \xi(S(\zeta_l, \zeta_l, \zeta_{l-1})), \quad (3.7)$$

which is a contradiction.

So max $\{S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l), S(\zeta_l, \zeta_l, \zeta_{l+1})\}$ is $S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l)$,
implies

$$\xi(S(\zeta_l, \zeta_{l+1}, \zeta_{l+1})0)) < \xi(S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l)) \text{ holds for all } l \in N \cup \{0\}. \quad (3.8)$$

So, the sequence $\{S(\zeta_l, \zeta_{l+1}, \zeta_{l+1})\}$ is nonnegative and nonincreasing. Now, we prove that $\{S(\zeta_l, \zeta_{l+1}, \zeta_{l+1})\} \to \varrho$ {and we claim $\varrho = 0$}. It is clear that $\{S(\zeta_l, \zeta_{l+1}, \zeta_{l+1})\}$ is a decreasing sequence. Therefore, there exists some positive number t such that $\lim_{n\to+\infty}\{S(\zeta_l, \zeta_{l+1}, \zeta_{l+1})\} = t$.
From 3.7 we have,

$$\frac{\xi(S(\zeta_{l+1}, \zeta_{n+2}, \zeta_{n+2}))}{\xi(S(\zeta_l, \zeta_{l+1}, \zeta_{l+1}))} \leq g(\xi(S(\zeta_l, \zeta_{l+1}, \zeta_{l+1}))) \leq 1.$$

Now taking limit $n \to +\infty$ we have $1 \leq g(\xi(S(\zeta_l, \zeta_{l+1}, \zeta_{l+1}))) \leq 1,$
that is,

$$g(\xi(S(\zeta_l, \zeta_{l+1}, \zeta_{l+1}))) = 1.$$

As $g \in$ G, we get $\lim_{n\to+\infty} \xi(S(\zeta_l, \zeta_{l+1}, \zeta_{l+1})) = 0$, that is

$$\lim_{n\to+\infty} S(\zeta_l, \zeta_{l+1}, \zeta_{l+1}) = 0. \tag{3.9}$$

Now, we present the sequence $\{\zeta_l\}$ is a Cauchy sequence. Suppose, however that $\{\zeta_l\}$ is not a Cauchy sequence. Then there exist $\epsilon > 0$ and sequences $\{\zeta_{m_k}\}$ and $\{\zeta_{l_k}\}$ such that, for all positive integers k, we have $m_\iota \geq m_\iota > k,$

$$S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_\iota}) \geq \epsilon. \tag{3.10}$$

In addition, in accordance with m_ι, we can choose m_ι in such a way that it is the smallest integer with $l_\iota \geq m_\iota$ and satisfies 3.10. Hence

$$S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{l_{k-1}}) < \epsilon. \tag{3.11}$$

Set $\delta_l = 2S(\zeta_l, \zeta_l, \zeta_{l-1})$. Using the lemma 2.4 and 2.5, we have

$$\begin{aligned}
\epsilon &\leq S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_\iota}) = S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_\iota}) \\
&\leq 2S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{l_{k-1}}) + S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{l_{k-1}}) \\
&\leq S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{l_{k-1}}) + \epsilon \\
&\leq \delta_{m_\iota} + \epsilon.
\end{aligned} \tag{3.12}$$

Letting $k \to +\infty$ in Equation 3.12 we derive that

$$\lim_{n\to\infty} S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_\iota}) = \epsilon. \tag{3.13}$$

Also, by Lemma 2.5 we obtain the following inequalities:

$$S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_\iota}) \leq 2S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{n_{k-1}}) + S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{l_{k-1}})$$
$$\leq 2S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{l_{k-1}}) + S(\zeta_{l_{k-1}}, \zeta_{l_{k-1}}, \zeta_{m_\iota})$$
$$= \delta_{m_\iota} + S(\zeta_{l_{k-1}}, \zeta_{l_{k-1}}, \zeta_{m_\iota}). \quad (3.14)$$

$$S(\zeta_{n_{k-1}}, \zeta_{n_{k-1}}, \zeta_{m_\iota}) \leq 2S(\zeta_{l_{k-1}}, \zeta_{l_{k-1}}, \zeta_{m_\iota}) + S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_\iota})$$
$$= \delta_{l_{k-1}} + S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_\iota}). \quad (3.15)$$

Letting $k \to \infty$ in Equation 3.15 and applying Equation 3.14 we get

$$\lim_{k\to+\infty} S(\zeta_{l_{k-1}}, \zeta_{l_{k-1}}, \zeta_{m_\iota}) = \epsilon,$$
$$S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{l_{k-1}}) = \epsilon. \quad (3.16)$$

Now, $\lim_{k\to+\infty}$

$$S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_\iota}) \leq 2S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_{k-1}}) + S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_{k-1}})$$
$$\leq 2S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_{k-1}}) + 2S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{n_{k-1}}) + S(\zeta_{m_{k-1}}, \zeta_{m_{k-1}}, \zeta_{l_{k-1}})$$
$$= \delta_{m_\iota} + \delta_{m_\iota} + S(\zeta_{m_{k-1}}, \zeta_{m_{k-1}}, \zeta_{l_{k-1}}). \quad (3.17)$$

$$S(\zeta_{m_{k-1}}, \zeta_{m_{k-1}}, \zeta_{l_{k-1}}) \leq 2S(\zeta_{m_{k-1}}, \zeta_{m_{k-1}}, \zeta_{m_\iota}) + S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{l_{k-1}})$$
$$\leq 2S(\zeta_{m_{k-1}}, \zeta_{m_{k-1}}, \zeta_{m_\iota}) + 2S(\zeta_{l_{k-1}}, \zeta_{l_{k-1}}, \zeta_{m_\iota}) + S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_\iota})$$
$$= \delta_{m_{k-1}} + \delta_{l_{k-1}} + S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_\iota}). \quad (3.18)$$

Letting $k \to \infty$ in Equation 3.18 and applying Equation 3.17 we get,

$$\lim_{k\to+\infty} S(\zeta_{m_{k-1}}, \zeta_{m_{k-1}}, \zeta_{l_{k-1}}) = \epsilon. \quad (3.19)$$

$$S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_\iota}) \leq 2S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_{k-1}}) + S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_{k-1}})$$
$$= \delta_{m_\iota} + S(\zeta_{m_{k-1}}, \zeta_{m_\iota}, \zeta_{m_\iota}). \quad (3.20)$$

$$S(\zeta_{m_{k-1}}, \zeta_{m_\iota}, \zeta_{m_\iota}) = S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_{k-1}})$$
$$S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_{k-1}}) \leq 2S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{l_{k-1}}) + S(\zeta_{l_{k-1}}, \zeta_{l_{k-1}}, \zeta_{m_{k-1}})$$
$$\leq 2S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{l_{k-1}}) + 2S(\zeta_{l_{k-1}}, \zeta_{l_{k-1}}, \zeta_{m_\iota}) + S(\zeta_{m_{k-1}}, \zeta_{m_{k-1}}, \zeta_{m_\iota})$$
$$\leq \delta_{m_\iota} + \delta_{l_{k-1}} + 2S(\zeta_{m_{k-1}}, \zeta_{m_{k-1}}, \zeta_{m_\iota}) + S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_\iota})$$
$$= \delta_{m_\iota} + \delta_{l_{k-1}} + \delta_{m_{k-1}} + S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_\iota}). \quad (3.21)$$

Letting $k \to \infty$ in Equation 3.21 and applying Equation 3.20 we get

$$\lim_{k\to+\infty} S(\zeta_{m_{k-1}}, \zeta_{m_\iota}, \zeta_{m_\iota}) = \epsilon. \tag{3.22}$$

$S(\zeta_{m_{k-1}}, \zeta_{m_{k-1}}, \zeta_{m_\iota}) = \delta_{m_{k-1}}$,

Letting $k \to \infty$, we obtain

$$\lim_{k\to+\infty} S(\zeta_{m_{k-1}}, \zeta_{m_{k-1}}, \zeta_{m_\iota}) = 0. \tag{3.23}$$

Consider Equation 3.6 with $\vartheta = \zeta_{m_\iota}$, $\zeta = \zeta_{m_{k-1}}$, $\nu = \zeta_{m_\iota}$, $\mu = \zeta_{l_{k-1}}$, $\vartheta^* = \zeta_{m_\iota}$,

$$S(\zeta_{m_{k-1}}, \zeta_{m_\iota}, \zeta_{m_\iota}) \leq g[(\Delta(\zeta_{m_{k-1}}, \zeta_{m_\iota}, \zeta_{l_{k-1}})][\Delta(\zeta_{m_{k-1}}, \zeta_{m_\iota}, \zeta_{l_{k-1}})], \tag{3.24}$$

where

$$\Delta(\zeta_{m_{k-1}}, \zeta_{m_\iota}, \zeta_{l_{k-1}}) = \max \Big\{ S(\zeta_{m_{k-1}}, \zeta_{m_{k-1}}, \zeta_{m_\iota}), S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{l_{k-1}}), S(\zeta_{l_{k-1}}, \zeta_{l_{k-1}}, \zeta_{m_{k-1}}),$$

$$\frac{S(\zeta_{m_{k-1}}, \zeta_{m_{k-1}}, \zeta_{m_\iota})S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{l_{k-1}})}{1 + S(\zeta_{m_{k-1}}, \zeta_{m_{k-1}}, \zeta_{m_\iota})S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{l_{k-1}})}, \frac{S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{l_{k-1}})S(\zeta_{l_{k-1}}, \zeta_{l_{k-1}}, \zeta_{m_{k-1}})}{1 + S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{l_{k-1}})S(\zeta_{l_{k-1}}, \zeta_{l_{k-1}}, \zeta_{m_{k-1}})},$$

$$\frac{S(\zeta_{m_{k-1}}, \zeta_{m_{k-1}}, \zeta_{m_\iota})S(\zeta_{l_{k-1}}, \zeta_{l_{k-1}}, \zeta_{m_{k-1}})}{1 + S(\zeta_{m_{k-1}}, \zeta_{m_{k-1}}, \zeta_{m_\iota})S(\zeta_{l_{k-1}}, \zeta_{l_{k-1}}, \zeta_{m_{k-1}})} \Big\},$$

$$\Delta(\zeta_{m_{k-1}}, \zeta_{m_\iota}, \zeta_{l_{k-1}}) = \max \Big\{ S(\zeta_{m_{k-1}}, \zeta_{m_{k-1}}, \zeta_{m_\iota}), S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{l_{k-1}}), S(\zeta_{l_{k-1}}, \zeta_{l_{k-1}}, \zeta_{m_{k-1}}) \Big\}. \tag{3.25}$$

Using the Equations 3.16,3.19,3.23 in 3.25 we obtain,

$$\delta_1(\zeta_{m_{k-1}}, \zeta_{m_\iota}, \zeta_{l_{k-1}}) = max\{0, \epsilon, \epsilon\}$$
$$= \epsilon. \tag{3.26}$$

Now taking limit $k \to \infty$ in Equation 3.24 and using Equations 3.2,3.26, we obtain,

$$\xi(\epsilon) \leq g(\xi(\epsilon)).\xi(\epsilon)\xi(\epsilon) = 1.$$

This contradicts itself by implying that $\epsilon = 0$. Hence,

$$\lim_{k\to+\infty} (S(\zeta_{m_\iota}, \zeta_{m_\iota}, \zeta_{m_{k+1}})) = 0. \tag{3.27}$$

Thus $\{\zeta_l\}$ is a Cauchy sequence. Since (B, S) is complete S - metric space, so there exists $\varrho \in B$ such that $\{\zeta_l\} \to \varrho$ as $l \to \infty$.

Conversely, for all $l \in N$, ,

$d_s(\varrho, C) \leq d_s(\varrho, T\zeta_l)$
$\leq d_s(\varrho, \zeta_{l+1}) + d_s(\zeta_{l+1}, T\zeta_l)$
$= d_s(\varrho, \zeta_{l+1}) + d_s(B, C)$. (3.28)

Taking limit as $l \to \infty$ in above inequality, we discover $\lim_{l\to\infty} d_s(\varrho, T\zeta_l) = d_s(\varrho, C) = d_s(\ , C)$.

Since C is approximatively compact with respect to B so the sequance $\{T\zeta_l\}$ has a subsequence $\{T\zeta_{m_l}\}$ that converges to some $\mu^* \in C$. Hence,B

$$d_s(\varrho, \mu^*) = \lim_{l\to\infty} d_s(\zeta_{l k+1}, T\zeta_{m_l}) = d_s(B, C), \quad (3.29)$$

and so $\varrho \in B_0$. Now since $T\varrho \in TB_0 \subseteq C_0$, so there exist $\kappa \in B_0$ such that

$$d_s(\kappa, T\varrho) = d_s(B, C).$$

By Equation 3.6 with $\vartheta = \zeta_{l+1}$, $\zeta = \zeta_l$, $\nu = \kappa$, $\mu = \varrho$, $\vartheta^* = \zeta_{n+2}$
we have

$$\xi(S(\zeta_{l+1}, \zeta_{l+2}, \kappa)) \leq g(\xi(\Delta(\zeta_l, \zeta_{l+1}, \varrho)))\xi(\Delta(\zeta_l, \zeta_{l+1}, \varrho)), \quad (3.30)$$

where

$$\Delta(\zeta_l, \zeta_{l+1}, \varrho) = max\{S(\zeta_l, \zeta_l, \zeta_{l+1}), S(\zeta_{l+1}, \zeta_{l+1}, \varrho), S(\varrho, \varrho, \zeta_l), \frac{S(\zeta_l, \zeta_l, \zeta_{l+1})S(\zeta_{l+1}, \zeta_{l+1}, \varrho)}{1 + S(\zeta_l, \zeta_l, \zeta_{l+1})S(\zeta_{l+1}, \zeta_{l+1}, \varrho)}, \frac{S(\zeta_{l+1}, \zeta_{l+1}, \varrho)S(\varrho, \varrho, \zeta_l)}{1 + S(\zeta_{l+1}, \zeta_{l+1}, \varrho)S(\varrho, \varrho, \zeta_l)}, \frac{S(\varrho, \varrho, \zeta_l)S(\zeta_l, \zeta_l, \zeta_{l+1})}{1 + S(\varrho, \varrho, \zeta_l)S(\zeta_l, \zeta_l, \zeta_{l+1})}\},$$

$$\Delta(\zeta_l, \zeta_{l+1}, \varrho) = max\{S(\zeta_l, \zeta_l, \zeta_{l+1}), S(\zeta_{l+1}, \zeta_{l+1}, \varrho), S(\varrho, \varrho, \zeta_l)\}.$$

Taking the limit $l \to \infty$

$$\lim_{l\to\infty} \Delta(\zeta_l, \zeta_{l+1}, \varrho) = \lim_{n\to\infty} max\{S(\zeta_l, \zeta_l, \zeta_{l+1}), S(\zeta_{l+1}, \zeta_{l+1}, \varrho, S(\varrho, \varrho, \zeta_l)\} = 0.$$

Taking the limit $l \to \infty$ in equation(3.28) and using $\lim_{l\to\infty} \Delta(\zeta_l, \zeta_{l+1}, \varrho) = 0$, we get

$$\xi(S(\varrho, \varrho, \kappa)) \leq g(\xi(0))\xi(0) = 0.$$

Then $S(\varrho, \varrho, \kappa) = 0$. That is $\varrho = \kappa$, so $d_s(\varrho, T\varrho) = d_s(B, C)$. Consequently, T has the "best proximity point".

Now we prove the uniqueness of "best proximity point" Suppose that p q such that $d_s(p, Tp) = d_s(B, C)$

and $d_s(q, \mathrm{T}q) = d_s(\mathrm{B}, \mathrm{C})$. Now by 3.6, with $\zeta = \vartheta = \vartheta^* = p$ and $\mu = \nu = q$ we get

$$\xi(S(p, p, q)) \leq g(\xi(\Delta(p, p, q)))\xi(\Delta(p, p, q)), \tag{3.31}$$

where

$$\begin{aligned}\Delta(p, p, q) &= \max\Big\{S(p, p, p), S(p, p, q), S(q, q, p), \frac{S(p, p, p)S(p, p, q)}{1 + S(p, p, p)S(p, p, q)},\\ &\qquad \frac{S(p, p, q)S(q, q, p)}{1 + S(p, p, q)S(q, q, p)}, \frac{S(q, q, p)S(p, p, p)}{1 + S(q, q, p)S(p, p, p)}\Big\}\\ &= \max\{S(p, p, q), S(q, q, p)\}.\end{aligned}$$

If $\max\{S(p, p, q), S(q, q, p)\} = S(p, p, q)$ then from Equation 3.31, we get

$$\begin{aligned}\xi(S(p, p, q)) &\leq g(\xi(S(p, p, q)))\xi(S(p, p, q)),\\ &< \xi(S(p, p, q))\end{aligned}$$

which is a contradiction. Thus $\max\{S(p, p, q), S(q, q, p)\} = S(q, q, p)$, again Equation 3.31 implies

$$\begin{aligned}\xi(S(p, p, q)) &\leq g(\xi(S(q, q, p)))\xi(S(q, q, p)),\\ &< \xi(S(q, q, p)).\end{aligned}$$

As ξ is non decreasing, then $q = p$.

Example 3.6. *Let* $\mathrm{X} = [0, +\infty)$. *It's simple to observe that* $S(\zeta, \mu, \varrho) = \frac{1}{8}(|\zeta - \varrho| + |\mu - \varrho|)$ *is an S-M on* X. *Then also, let* $d_s(\mathrm{B}, \mathrm{C}) = \frac{1}{2}|\zeta - \mu|$. *Let* $\mathrm{B} = \{1, 2, 3, 4\}$ *and* $\mathrm{C} = \{6, 7, 8, 9\}$ *Define* $\mathrm{T} : \mathrm{B} \to \mathrm{C}$

$$\mathrm{T} = \begin{cases} 6 & \zeta = 4, \\ \zeta + 4 & \textit{otherwise.} \end{cases}$$

Also define ,

$$\alpha(\vartheta, \nu, \kappa) = \begin{cases} 1 & \textit{if } \vartheta, \nu, \kappa \in \mathrm{B}, \\ 0 & \textit{otherwise.} \end{cases}$$

Also consider $g : [0, +\infty) \to [0, 1)$ *and* $\xi : [0, \infty) \to [0, \infty)$ *defined by* $\xi(\zeta) = \zeta, g(\zeta) = \frac{\zeta}{2}$ *respectively. Clearly* $d_s(\mathrm{B}, \mathrm{C}) = 1$, $\mathrm{B}_0 = \{4\}$, $\mathrm{C}_0 = \{6\}$ *and* $\mathrm{T}(\mathrm{B}_0) \subseteq \mathrm{T}(\mathrm{C}_0)$. *Let* $d_s(\vartheta, \mathrm{T}\zeta) = d_s(\mathrm{B}, \mathrm{C})$ *and* $d_s(\nu, \mathrm{T}\mu) = d_s(\mathrm{B}, \mathrm{C}) = 1$. *Then* $(\vartheta, \zeta), (\nu, \mu) \in \{(4, 4), (4, 2)\}$. *Also, if* $d_s(\vartheta^*, \mathrm{T}\vartheta) = d_s(\mathrm{B}, \mathrm{C}) = 1$, *then* $\vartheta^* = 4$. *Therefore, if*

$$\begin{aligned}d_s(\vartheta, \mathrm{T}\zeta) &= d_s(\mathrm{B}, \mathrm{C}),\\ d_s(\vartheta^*, \mathrm{T}\vartheta) &= d_s(\mathrm{B}, \mathrm{C}),\\ d_s(\nu, \mathrm{T}\mu) &= d_s(\mathrm{B}, \mathrm{C}),\end{aligned}$$

then

$$(\vartheta, \vartheta^*, \nu, \zeta, \mu) \in \{(4, 4, 4, 4, 4), (4, 4, 4, 2, 2), (4, 4, 4, 2, 4), (4, 4, 4, 4, 2)\}.$$

Now $\vartheta = \vartheta^* = \nu = 4$ *so,* $\xi(S(\vartheta, \vartheta^*, \nu)) = 0$. *Hence,*

$$\xi(S(\vartheta, \vartheta^*, \nu)) = 0 \leq \frac{1}{2}x \leq g(\xi(\Delta(\zeta, \vartheta, \mu)))\xi(\Delta(\zeta, \vartheta, \mu)),$$

where

$$\Delta(\zeta, \vartheta, \mu) = \max\left\{S(\zeta, \zeta, \vartheta), S(\vartheta, \vartheta, \mu), S(\mu, \mu, \zeta), \frac{S(\zeta, \zeta, \vartheta)S(\vartheta, \vartheta, \mu)}{1 + S(\zeta, \zeta, \vartheta)S(\vartheta, \vartheta, \mu)}, \frac{S(\vartheta, \vartheta, \mu)S(\mu, \mu, \zeta)}{1 + S(\vartheta, \vartheta, \mu)S(\mu, \mu, \zeta)}, \frac{S(\mu, \mu, \zeta)S(\zeta, \zeta, \vartheta)}{1 + S(\mu, \mu, \zeta)S(\zeta, \zeta, \vartheta)}\right\}.$$

Let $\zeta = 2, \vartheta = 1, \mu = 4$, *we obtained*

$$\Delta(2, 1, 4) = \max\left\{S(2, 2, 1), S(1, 1, 4), S(4, 4, 2), \frac{S(2, 2, 1)S(1, 1, 4)}{1 + S(2, 2, 1)S(1, 1, 4)}, \frac{S(1, 1, 4)S(4, 4, 1)}{1 + S(1, 1, 4)S(T4, 4, 1)}, \frac{S(4, 4, 1)S(2, 2, 1)}{1 + S(4, 4, 1)S(2, 2, 1)}\right\}$$

$$= \max\left\{\frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{3}{19}, \frac{3}{11}, \frac{1}{9}\right\} = \frac{3}{4}.$$

Thus T *is a generalized rational* α_s*–Proximal contraction mapping. All the conditions of Theorem 3.2 are true and T has a unique best proximity point. Here,* $\varrho = 4$ *is the unique best proximity point in* T

If in Theorem 3.2 we take $\xi(s) = s$, $g(t) = t^r$ where $0 < r < 1$ and $r \in (0, \infty)$ then we deduce the following corollary.

Corollary 3.6.1. *Suppose* B, C *be two non-empty subsets of a S-M space* (X , S) *such that* (B, S) *is a complete S-M space,* B_0 *is non-empty, and* C *is approximatively compact with respect to* B. *Assume that* T : B → C *is a non-self-mapping such that* $T(B_0) \subseteq C_0$ *and, for* $\zeta, \mu, \vartheta, \vartheta^*, \nu \in B$

$$\left.\begin{array}{l} d_s(\vartheta, T\zeta) = d_s(B, C), \\ d_s(\vartheta^*, T\vartheta) = d_s(B, C), \\ d_s(\nu, T\mu) = d_s(B, C), \end{array}\right\} \Longrightarrow \alpha_s(\vartheta, \vartheta^*, \nu)S(\vartheta, \nu, \kappa) \leq \Delta(\zeta, \vartheta, \mu)^r\Delta(\zeta, \vartheta, \mu)$$

holds where $0 < r < 1$.

and $$\Delta(\zeta, \vartheta, \mu) = \max\left\{S(\zeta, \zeta, \vartheta), S(\vartheta, \vartheta, \mu), S(\mu, \mu, \zeta), \frac{S(\zeta, \zeta, \vartheta)S(\vartheta, \vartheta, \mu)}{1 + S(\zeta, \zeta, \vartheta)S(\vartheta, \vartheta, \mu)}, \frac{S(\vartheta, \vartheta, \mu)S(\mu, \mu, \zeta)}{1 + S(\vartheta, \vartheta, \mu)S(\mu, \mu, \zeta)}, \frac{S(\mu, \mu, \zeta)S(\zeta, \zeta, \vartheta)}{1 + S(\mu, \mu, \zeta)S(\zeta, \zeta, \vartheta)}\right\}.$$

Then T *has unique best proximity point, that is, there exists unique* $\varrho \in B$ *such that* $d_s(\varrho, T\varrho) = d_s(B, C)$If in

Theorem 3.2 we take $\xi(s) = s$, $g(t) = \frac{1}{1+t}$ then we conclude the following corollary.

Corollary 3.6.2. *Suppose* B, C *be two non-empty subsets of an S-M space* (X, S) *such that* (B, S) *is a complete S-M space,* B_0 *is non-empty, and* C *is approximatively compact with respect to* B. *Assume that*

$T : B \to C$ *is a non-self-mapping such that* $T(B_0) \subseteq C_0$ *and for* $\zeta, \mu, \vartheta, \vartheta^*, \nu \in B$

$$\left.\begin{array}{l} d_s(\vartheta, T\zeta) = d_s(B, C), \\ d_s(\vartheta^*, T\vartheta) = d_s(B, C), \\ d_s(\nu, T\mu) = d_s(B, C), \end{array}\right\} \Longrightarrow \alpha_s(\vartheta, \vartheta^*, \nu)S(\vartheta, \vartheta^*, \nu) \le \frac{1}{1+\Delta(\zeta,\vartheta,\mu)} \Delta(\zeta, \vartheta, \mu)$$

where $\Delta(\zeta, \vartheta, \mu) = \max\left\{S(\zeta, \zeta, \vartheta), S(\vartheta, \vartheta, \mu), S(\mu, \mu, \zeta), \frac{S(\zeta, \zeta, \vartheta)S(\vartheta, \vartheta, \mu)}{1 + S(\zeta, \zeta, \vartheta)S(\vartheta, \vartheta, \mu)}, \frac{S(\vartheta, \vartheta, \mu)S(\mu, \mu, \zeta)}{1 + S(\vartheta, \vartheta, \mu)S(\mu, \mu, \zeta)}, \frac{S(\mu, \mu, \zeta)S(\zeta, \zeta, \vartheta)}{1 + S(\mu, \mu, \zeta)S(\zeta, \zeta, \vartheta)}\right\}.$

holds for $0 \le r < 1$*. Then T has unique best proximity point that is, there exists unique* $\varrho \in B$ *such that* $d_s(\varrho, T\varrho) = d_s(B, C)$.

In Theorem 3.2 we can have another result.

Let (X, S) be a S-M space, and let $\alpha_s, \eta_s : B \times B \times B \to [0, +\infty)$ be a function. Mapping $T : B \to C$ is called generalized rational α_s−Proximal contraction type mapping with respect to η_s if there exist $g \in G$ such that, for all $\zeta, \vartheta, \vartheta^*, \mu, \nu \in B$.
$\alpha_s(\vartheta, \vartheta^*, \nu) \ge \eta_s(\vartheta, \vartheta^*, \nu)$

$\Longrightarrow S(\vartheta, \vartheta^*, \nu) \le g(\xi(\Delta(\zeta, \vartheta, \mu)))\xi(\Delta(\zeta, \vartheta, \mu))$ where,

$$\Delta(\zeta, \vartheta, \mu) = \max\left\{S(\zeta, \zeta, \vartheta), S(\vartheta, \vartheta, \mu), S(\mu, \mu, \zeta), \frac{S(\zeta, \zeta, \vartheta)S(\vartheta, \vartheta, \mu)}{1 + S(\zeta, \zeta, \vartheta)S(\vartheta, \vartheta, \mu)}, \frac{S(\vartheta, \vartheta, \mu)S(\mu, \mu, \zeta)}{1 + S(\vartheta, \vartheta, \mu)S(\mu, \mu, \zeta)}, \frac{S(\mu, \mu, \zeta)S(\zeta, \zeta, \vartheta)}{1 + S(\mu, \mu, \zeta)S(\zeta, \zeta, \vartheta)}\right\}.$$

Theorem 3.7. *Let* (X, S) *be a CS-M space. Let* T *be an* α_s*−Proximal admissible mapping with respectto* η_s *such that the following hold:*

1. T *is a generalized rational* α_s*− Proximal contraction type mapping.*
2. *There exists* $\zeta_0 \in X$ *such that* $\alpha_s(\zeta_0, \zeta_0, T\zeta_0) \ge \eta_s(\zeta_0, \zeta_0, T\zeta_0)$.
3. *This continuous.*
4. *If* $\{\zeta_l\}$ *is a sequence in* X *such that* $\alpha_s(\zeta_l, \zeta_l, \zeta_{l+1}) \ge \eta_s(\zeta_l, \zeta_l, \zeta_{l+1})$ *for all* $l \in N \cup \{0\}$ *and* $\zeta_l \to \varrho \in B$ *as* $l \to +\infty$*, then there exists a subsequence* $\{\zeta_{m_{l}}\}$ *of* $\{\zeta_l\}$ *such that* $\alpha_s(\zeta_{m_{l}}, \zeta_{m_{l}}, \varrho) \ge \eta_s(\zeta_{m_{l}}, \zeta_{m_{l}}, \varrho)$ *for all k.*

Then T *has best proximity point.*

Proof. Since subset B_0 is not empty, we take ζ_0 in B_0. Taking $T\zeta_0 \in T(B_0) \subseteq C_0$ into account, we can find $\zeta_1 \in B_0$ such that

$$d_s(\zeta_1, T\zeta_0) = d_s(B, C).$$

Further, since $T\zeta_1 \in T(B_0) \subseteq C_0$, it follows that there are element ζ_2 and ζ_3 in B_0 such that

$$d_s(\zeta_2, T\zeta_1) = d_s(B, C),$$
$$d_s(\zeta_3, T\zeta_2) = d_s(B, C).$$

Recursively, we obtain a sequence $\{\zeta_l\}$ in B_0 satisfying

$$d_s(\zeta_{l+1}, T\zeta_l) = d_s(B, C), \forall l \in N \cup \{0\}.$$

By taking $\vartheta = \zeta_l$, $\zeta = \zeta_{l-1}$, $\nu = \zeta_{l+1}$, $\mu = \zeta_l$, $\vartheta^* = \zeta_{l+1}$, Equation 3.2 gives

$$\alpha_s(\zeta_l, \zeta_{l+1}, \zeta_{l+1})\xi(S(\zeta_l, \zeta_{l+1}, \zeta_{l+1})) \leq g(\xi(\Delta(\zeta_{l-1}, \zeta_l, \zeta_l)))(\xi(\Delta(\zeta_{l-1}, \zeta_l, \zeta_l)). \quad (3.32)$$

By condition (3), we have $\alpha_s(\zeta_0, \zeta_1, \zeta_1) \geq \eta_s(\zeta_0, \zeta_1, \zeta_1)$

$$\eta_s(\zeta_l, \zeta_{l+1}, \zeta_{l+1})\xi(S(\zeta_l, \zeta_{l+1}, \zeta_{l+1})) \leq g(\xi(\Delta(\zeta_{l-1}, \zeta_l, \zeta_l)))(\xi(\Delta(\zeta_{l-1}, \zeta_l, \zeta_l)).$$

By the assumption $\eta_s(\zeta_0, \zeta_1, \zeta_1) \geq 1$ and T is α_s– Proximal admissible, we have $\eta_s(\zeta_l, \zeta_{l+1}, \zeta_{l+1}) \geq 1$ for all $l \in N \cup \{0\}$.

$$\xi(S(\zeta_l, \zeta_{l+1}, \zeta_{l+1})) \leq g(\xi(\Delta(\zeta_{l-1}, \zeta_l, \zeta_l)))\xi(\Delta(\zeta_{l-1}, \zeta_l, \zeta_l)) \quad (3.33)$$

where

$$\Delta(\zeta_{l-1}, \zeta_l, \zeta_l) = \max \Big\{ S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l), S(\zeta_l, \zeta_l, \zeta_l), S(\zeta_l, \zeta_l, \zeta_{l-1}), \frac{S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l)S(\zeta_l, \zeta_l, \zeta_l)}{1 + S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l)S(\zeta_l, \zeta_l, \zeta_l)}, \frac{S(\zeta_l, \zeta_l, \zeta_l)S(\zeta_l, \zeta_l, \zeta_{l-1})}{1 + S(\zeta_l, \zeta_l, \zeta_l)S(\zeta_l, \zeta_l, \zeta_{l-1})}, \frac{S(\zeta_l, \zeta_l, \zeta_{l-1})S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l)}{1 + S(\zeta_l, \zeta_l, \zeta_{l-1})S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l)} \Big\}$$
$$= \max\{S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l), S(\zeta_l, \zeta_l, \zeta_{l-1})\}.$$

If $\max \{S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l), S(\zeta_l, \zeta_l, \zeta_{l-1})\} = S(\zeta_l, \zeta_l, \zeta_{l-1})$ then the Equation 3.33 becomes

$$\xi(S(\zeta_l, \zeta_{l+1}, \zeta_{l+1})) \leq g(\xi(S(\zeta_l, \zeta_l, \zeta_{l-1})))\xi(S(\zeta_l, \zeta_l, \zeta_{l-1}))$$
$$< \xi(S(\zeta_l, \zeta_l, \zeta_{l-1})), \quad (3.34)$$

which is a contradiction.

So $\max \{S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l), S(\zeta_l, \zeta_l, \zeta_{l+1})\}$ is $S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l)$.
This implies

$$\xi(S(\zeta_l, \zeta_{l+1}, \zeta_{l+1})0)) < \xi(S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l)) \text{ holds for all } l \in N \cup \{0\}. \quad (3.35)$$

In a similar way Theorem 3.2, we can prove that T has a best proximity point.

Theorem 3.8. *Let* B, C *be two non-empty subsets of an S-M space* (X, S) *such that* (B, S) *is a complete S-M space,* B_0 *is non-empty, and* C *is approximatively compact with respect to* B. *Assume that* $T : B \to C$

is a non-self-mapping such tha $T(B_0) \subseteq C_0$ *and, for* $\zeta, \mu, \vartheta, \vartheta^*, \nu \in B$

$$\left.\begin{array}{l} d_s(\vartheta, T\zeta) = d_s(B, C), \\ d_s(\vartheta^*, T\vartheta) = d_s(B, C), \\ d_s(\nu, T\mu) = d_s(B, C), \end{array}\right\} \Longrightarrow \begin{array}{l} S(\vartheta, \vartheta^*, \nu) \leq \alpha S(\zeta, \zeta, \vartheta) + \beta \sqrt{\dfrac{S(\zeta,\zeta,\vartheta)S(\zeta,\zeta,\mu)}{1+S(\vartheta^*,\vartheta^*,\zeta)}} \\ +\gamma S(\mu, \mu, \zeta) + \delta \dfrac{S(\mu,\mu,\zeta)}{1+S(\zeta,\zeta,\vartheta)} \end{array} \quad (3.36)$$

holds where $\alpha, \beta, \gamma, \delta \geq 0$ *and* $\alpha + \beta + \gamma + \delta < 1$. *Then* T *has the unique best proximity point.*

Proof. Following the same lines in the proof of Theorem 3.2, we can construct a sequences $\{\zeta_l\}$ in B_0 satisfying

$$d_s(\zeta_{l+1}, T\zeta_n) = d_s(B, C); \forall l \in \mathbb{N} \cup \{0\}.$$

From (3.36) with $\zeta = \zeta_{l-1}$, $\vartheta = \zeta_l$, $\mu = \zeta_l$, $\nu = \zeta_{l+1}$, $\vartheta^* = \zeta_{l+1}$, we obtain

$$\begin{aligned} S(\zeta_l, \zeta_{l+1}, \zeta_{l+1}) &\leq \alpha S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l) + \beta \sqrt{\frac{S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l)S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l)}{1 + S(\zeta_{l+1}, \zeta_{l+1}, \zeta_{l-1})}} \\ &+ \gamma S(\zeta_l, \zeta_l, \zeta_{l-1}) + \delta \frac{S(\zeta_l, \zeta_l, \zeta_{l-1})}{1 + S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l)} \\ &= \left(\alpha + \frac{\beta}{1 + S(\zeta_{l+1}, \zeta_{l+1}, \zeta_{l-1})} + \gamma + \frac{\delta}{1 + S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l)}\right) S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l) \\ &\leq (\alpha + \beta + \gamma + \delta) S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l), \end{aligned}$$

for all $l \in \mathbb{N} \cup \{0\}$. This implies

$$S(\zeta_l, \zeta_{l+1}, \zeta_{l+1}) \leq k^l S(\zeta_0, \zeta_0, \zeta_1), \quad (3.37)$$

where $k = \alpha + \beta + \gamma + \delta < 1$. Now, for all $m, l \in \mathbb{N}$, $n < m$, by Lemma 2.4 and Equation 3.36, we have

$$\begin{aligned} S(\zeta_l, \zeta_m, \zeta_m) &\leq 2S(\zeta_l, \zeta_l, \zeta_{l+1}) + S(\zeta_m, \zeta_m, \zeta_{l+1}) \\ &= 2S(\zeta_l, \zeta_l, \zeta_{l+1}) + S(\zeta_{l+1}, \zeta_{l+1}, \zeta_m) \\ &\leq 2k^n S(\zeta_0, \zeta_0, \zeta_1) + 2S(\zeta_{l+1}, \zeta_{l+1}, \zeta_{l+2}) + S(\zeta_m, \zeta_m, \zeta_{l+2}) \\ &= 2k^n S(\zeta_0, \zeta_0, \zeta_1) + 2S(\zeta_{l+1}, \zeta_{l+1}, \zeta_{l+2}) + S(\zeta_{l+2}, \zeta_{l+2}, \zeta_m) \\ &\vdots \\ &\leq 2[k^l + \ldots\ldots + k^{m-1}] S(\zeta_0, \zeta_0, \zeta_1) \\ &\leq \frac{2k^l}{1-k} S(\zeta_0, \zeta_0, \zeta_1). \end{aligned}$$

Taking limit as $n, m \to \infty$, we get $S(\zeta_l, \zeta_l, \zeta_m) \to 0$. This gives that $\{\zeta_l\}$ is a Cauchy sequence in S-M space (X, S). Due to the completeness of (B, S), there exists $\varrho \in B$ such that $\{\zeta_l\}$ converges to ϱ. As in the proof of Theorem 3.2, we have $d_s(\kappa, T\varrho) = d_s(B, C)$ for some $\kappa \in B_0$. From Equation 3.36 with

$\zeta = \zeta_{l-1}, \vartheta = \zeta_l, \vartheta^* = \zeta_{l+1}, \mu = \varrho$ and $\nu = \kappa$, we deduce

$$S(\zeta_l, \zeta_{l+1}, \kappa) \leq \alpha S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l) + \beta \frac{\sqrt{S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l) S(\zeta_{l-1}, \zeta_{l-1}, z)}}{1 + S(\zeta_{l+1}, \zeta_{l+1}, \zeta_{l-1})} + \gamma S(\varrho, \varrho, \zeta_{l-1}) + \delta \frac{S(\varrho, \varrho, \zeta_{l-1})}{1 + S(\zeta_{l-1}, \zeta_{l-1}, \zeta_l)}.$$

By taking limit as $l \to \infty$ in the inequality mentioned above, we obtain $S(\varrho, \varrho, \kappa) = 0$; that is $\varrho = \kappa$. Hence, $d_s(\varrho, T\varrho) = d_s(\kappa, T\varrho) = d_s(B, C)$; that is, T has the best proximity point. To prove uniqueness, suppose that $p \neq q$, $d_s(p, Tp) = d_s(B, C)$ and $d_s(q, Tq) = d_s(B, C)$. Now by Equation 3.36 with $\zeta = \vartheta = \vartheta^* = p$ and $\mu = \nu = q$ we have,

$$\begin{aligned} S(p, p, q) &\leq \alpha S(p, p, p) + \beta \frac{\sqrt{S(p, p, p) S(p, p, q)}}{1 + S(p, p, p)} + \gamma S(q, q, p) + \delta \frac{S(q, q, p)}{1 + S(p, p, p)} \\ &\leq (\gamma + \delta) S(q, q, p) \\ &= (\gamma + \delta) S(p, p, q), \end{aligned}$$

which implies $S(p, p, q) = 0$. Hence $p = q$, that is T has the unique best proximity point.

By taking $\beta = \gamma = \delta = 0$ in Theorem (3.5), we obtain the following Corollary:

Corollary 3.8.1. *Suppose* B, C *be two non-empty subsets of an S-M space* (X , S) *such that* (B, S) *is a complete S-M space,* B_0 *is non-empty, and* C *is approximatively compact with respect to* B. *Assume that* $T : B \to C$ *is a non-self-mapping such that* $T(B_0) \subseteq C_0$ *and, for* $\zeta, \mu, \vartheta, \nu \in B$

$$\left.\begin{aligned} d_s(\vartheta, T\zeta) &= d_s(B, C), \\ d_s(\vartheta^*, T\vartheta) &= d_s(B, C), \\ d_s(\nu, T\mu) &= d_s(B, C), \end{aligned}\right\} \Longrightarrow \quad S(\vartheta, \vartheta^*, \nu) \leq \alpha S(\zeta, \zeta, \vartheta)$$

holds where $0 \leq \alpha < 1$. *Then* T *has the unique best proximity point.*

4 Application to Fixed Point Theory

In this section, as an application of our best proximity results, we will derive certain new fixed point results

Note that if

$$\left.\begin{aligned} d_s(\vartheta, T\zeta) &= d_s(B, C), \\ d_s(\vartheta^*, T\vartheta) &= d_s(B, C), \\ d_s(\nu, T\mu) &= d_s(B, C), \end{aligned}\right\} \Longrightarrow \quad \alpha_s(\vartheta, \vartheta^*, \nu)\xi(S(\vartheta, \vartheta^*, \nu)) \leq g(\xi(\Delta(\zeta, \vartheta, \mu)))\xi(\Delta(\zeta, \vartheta, \mu)), \tag{4.1}$$

where

$$\Delta(\zeta,\vartheta,\mu)=\max\left\{S(\zeta,\zeta,\vartheta),S(\vartheta,\vartheta,\mu),S(\mu,\mu,\zeta),\frac{S(\zeta,\zeta,\vartheta)S(\vartheta,\vartheta,\mu)}{1+S(\zeta,\zeta,\vartheta)S(\vartheta,\vartheta,\mu)},\frac{S(\vartheta,\vartheta,\mu)S(\mu,\mu,\zeta)}{1+S(\vartheta,\vartheta,\mu)S(\mu,\mu,\zeta)},\frac{S(\mu,\mu,\zeta)S(\zeta,\zeta,\vartheta)}{1+S(\mu,\mu,\zeta)S(\zeta,\zeta,\vartheta)}\right\} \quad (4.2)$$

and B = C = X, then $\vartheta = T\zeta$, $\vartheta^* = T\vartheta$, and $\nu = T\mu$. That is, $\vartheta^* = T^2\zeta$. Therefore, if in Theorem 3.5 wetake B = C = X, we deduce the following recent result.

Theorem 4.1. *Let* B *be non-empty subsets of an S-M space* (X,S) *such that* (B,S) *be a complete S-Mspace and* B_0 *be non-empty set.* B *is approximatively compact with respect to* B.

1. T *is a generalized rational* α_s*–Proximal contraction mapping.*
2. *There exists* $\zeta_0 \in B$ *such that* $\alpha_s(\zeta_0, \zeta_1, T\zeta_1) \geq 1$.
3. T *is continuous.*

Then T *has a fixed point* $\varrho \in B$ *, and* T *is a Picard operator, that is,* $\{T^n\zeta_0\}$ *converges to a.*

Theorem 4.2. *Let* B *be non-empty subsets of an S-M space* (X,S) *such that* (B,S) *be a complete S-Mspace and* B_0 *be non-empty set.* B *is approximatively compact with respect to* B.

1. T *is a generalized rational* α_s*–Proximal contraction mapping.*
2. *There exists* $\zeta_0 \in B$ *such that* $\alpha_s(\zeta_0, \zeta_1, T\zeta_1) \geq 1$.
3. T *is continuous.*
4. *If* $\{\zeta_l\}$ *is a sequence in* B *such that* $\alpha_s(\zeta_l, \zeta_{l+1}, \zeta_{l+1}) \geq 1$ *for all* $l \in N \cup \{0\}$ *and* $\zeta_l \to \varrho \in B$ *as* $l \to +\infty$*, then there exists a subsequence* $\{\zeta_{m_l}\}$ *of* $\{\zeta_n\}$ *such that* $\alpha_s(\zeta_{m_l}, \varrho, \varrho) \geq 1$ *for all k.*

Then T *has a fixed point* $\varrho \in B$ *, and* T *is a Picard operator, that is,* $\{T^n\zeta_0\}$ *converges to a.*

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