# Some New Results on Energy of Graphs with Self Loops

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# Some New Results on Energy of Graphs with Self Loops

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**Abstract**

The graph $\boldsymbol{G_\sigma}$ is obtained from graph $\boldsymbol{G}$ by attaching self loops on $\boldsymbol{\sigma}$ vertices. The energy $\boldsymbol{E(G_\sigma)}$ of the graph $\boldsymbol{G_\sigma}$ with order $\boldsymbol{n}$ and eigenvalues $\boldsymbol{\lambda_1, \lambda_2, \ldots, \lambda_n}$ is defined as $\boldsymbol{E(G_\sigma) = \sum_{i=1}^{n} \left|\lambda_i - \frac{\sigma}{n}\right|}$. It has been proved that if $\boldsymbol{\sigma = 0 \ or \ n}$ then $\boldsymbol{E(G) = E(G_\sigma)}$. The obvious question arise: Are there any graph such that $\boldsymbol{E(G) = E(G_\sigma)}$ and 0< $\boldsymbol{\sigma < n}$? We have found an affirmative answer of this question and contributed a graph family which satisfies this property.

**Keywords:** Eigenvalue, Energy, Self-loops

## 1 Introduction

For standard terminology and notations related to graph theory, we follow Balakrishnan and Ranganathan [9] while any terms related to algebra we depend on Lang [11].

An undirected graph without multiple edges and self-loops is called a simple graph. The adjacency matrix $A(G)$ of a simple graph $G$ with vertex set

$\{v_1, v_2, \ldots, v_n\}$ is $n$-ordered symmetric matrix $A(G) = [a_{ij}]$ such that,

$$a_{ij} = \begin{cases} 1 & \text{if the vertex } v_i \text{ is adjacent with vertex } v_j, \\ 0 & \text{if the vertex } v_i \text{ is not adjacent with vertex } v_j. \end{cases}$$

The characteristic polynomial of the adjacency matrix $A(G)$ is denoted by $\phi(G : x)$. The roots of characteristic polynomial $\lambda_1, \lambda_2, \ldots, \lambda_n$ are called the eigenvalues of graph $G$. The energy $E(G)$ of graph $G$ is developed by Gutman [4] in 1978 as $E(G) = \sum_{i=1}^{n} |\lambda_i|$.

This graph energy is an emerging subject for a researchers of applied mathematics and mathematical chemistry. A brief account of graph energy of simple graphs can be found in [3, 8, 10] as well as in the books [2, 13]. The variants of graph energy can be found in [1, 7, 12].

Recently the concept of energy of graphs with self-loops is open-up by Gutman *et al.* [5]. Let $G_\sigma$ be the graph obtained from graph $G$ by attaching self loops on $\sigma$ chosen vertices. The adjacency matrix $A(G_\sigma)$ of graph $G_\sigma$ is an $n \times n$ symmetric matrix such that $A(G_\sigma) = A(G) + I_\sigma$, where $I_\sigma$ is a square matrix of order $n$ with exactly $\sigma$ ones on the main diagonal and all other entries are zero. The eigenvalues of $A(G_\sigma)$ are denoted by $\lambda_1(G_\sigma), \lambda_2(G_\sigma), \cdots, \lambda_n(G_\sigma)$. The energy $E(G_\sigma)$ of $G_\sigma$ is

$$E(G_\sigma) = \sum_{i=1}^{n} \left| \lambda_i(G_\sigma) - \frac{\sigma}{n} \right|$$

Gutman *et al.* have [5] conjectured that for any graph $G$ of order $n$, $E(G) < E(G_\sigma)$. Irena *et al.* [6] have disproved this conjuncture by showing examples of graphs such that $E(G) > E(G_\sigma)$. It has been shown that [5] if $\sigma = 0$ *or* $n$ then $E(G) = E(G_\sigma)$. In the present paper we have obtained a graph family such that $E(G) = E(G_\sigma)$ and $0 < \sigma < n$.

## 2 Main Results

**Theorem 1** *Let $G$ be the simple graph of order $n$ with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and $G^l$ be the graph obtained from $G$ by adding a loop on each vertex of $G$ then $E((G \cup G^l)_n) = 2E(G)$, if $|\lambda_i| \geq \frac{1}{2}$, for each $i = 1, 2, \cdots, n$.*

***Proof***: Let $H_n = G \cup G^l$. The graph $H_n$ contains $2n$ vertices and $n$ loops. The adjacency matrix of $H$ is given by:

$$A(H_n) = \begin{bmatrix} A(G) & 0 \\ 0 & A(G) + I_n \end{bmatrix}$$

The characteristic polynomial of above matrix is given by:

$$\phi(H_n : x) = \begin{vmatrix} xI - A(G) & 0 \\ 0 & xI - (A(G) + I_n) \end{vmatrix}$$

It follows that if $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of $A$ then,

$$\phi(H_n : x) = \prod_{i=1}^{n} (x - \lambda_i)(x - (\lambda_i + 1))$$

The roots of above characteristic polynomial are:

$$x = \lambda_i, x = \lambda_i + 1$$

, for each $i = 1, 2, \cdots, n$
Here,

$$\begin{aligned} E(H_n) &= \sum_{i=1}^{n} \left( \left| \lambda_i - \frac{n}{2n} \right| + \left| \lambda_i + 1 - \frac{n}{2n} \right| \right) \\ &= \sum_{i=1}^{n} \left( \left| \lambda_i - \frac{1}{2} \right| + \left| \lambda_i + \frac{1}{2} \right| \right) \end{aligned} \tag{1}$$

Suppose that $|\lambda_i| \geq \frac{1}{2}$, for all $1 \leq i \leq n$ then

$$\left| \lambda_i - \frac{1}{2} \right| = \begin{cases} |\lambda_i| - \frac{1}{2}, & \text{if } \lambda_i \geq 0 \\ |\lambda_i| + \frac{1}{2}, & \text{if } \lambda_i < 0 \end{cases}$$

and

$$\left| \lambda_i + \frac{1}{2} \right| = \begin{cases} |\lambda_i| + \frac{1}{2}, & \text{if } \lambda_i \geq 0 \\ |\lambda_i| - \frac{1}{2}, & \text{if } \lambda_i < 0 \end{cases}$$

Therefore, from equation 1

$$\begin{aligned} E(H_n) &= \sum_{i=1}^{n} \left( \left| \lambda_i - \frac{1}{2} \right| + \left| \lambda_i + \frac{1}{2} \right| \right) \\ &= \sum_{\lambda_i \geq 0} \left( \left| \lambda_i - \frac{1}{2} \right| + \left| \lambda_i + \frac{1}{2} \right| \right) + \sum_{\lambda_i < 0} \left( \left| \lambda_i - \frac{1}{2} \right| + \left| \lambda_i + \frac{1}{2} \right| \right) \\ &= \sum_{\lambda_i \geq 0} \left( |\lambda_i| - \frac{1}{2} + |\lambda_i| + \frac{1}{2} \right) + \sum_{\lambda_i < 0} \left( |\lambda_i| + \frac{1}{2} + |\lambda_i| - \frac{1}{2} \right) \end{aligned}$$

$$= 2\left(\sum_{\lambda_i \geq 0} |\lambda_i| + \sum_{\lambda_i < 0} |\lambda_i|\right)$$
$$= 2\sum_{i=1}^{n} |\lambda_i|$$
$$= 2E(G)$$

*Example 1* We now given an example of graph $G$ such that $E(G) = E(G_\sigma)$ and $0 < \sigma < n$. Consider the graph $H = K_3 \cup K_3$ and $H_3 = K_3 \cup K_3^l$. The graph $H_3$ contains 6 vertices and three loops. It is known fact that $E(K_3) = 4$ and hence $E(H) = E(K_3 \cup K_3) = 2E(G) = 2(4) = 8$.

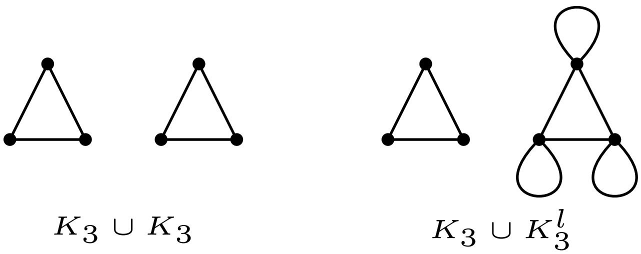


**Fig. 1**

The adjacency matrix of $H_3$ is:

$$A(H_3)=\begin{bmatrix} 0&1&1&0&0&0\\ 1&0&1&0&0&0\\ 1&1&0&0&0&0\\ 0&0&0&1&1&1\\ 0&0&0&1&1&1\\ 0&0&0&1&1&1 \end{bmatrix}$$

The eigenvalues of $K_3^l$ are $3^1, 2^1, (-1)^2$ and $0^2$.
Hence,

$$E(H_3) = \left|3 - \frac{3}{6}\right| + \left|2 - \frac{3}{6}\right| + 2\left|-1 - \frac{3}{6}\right| + 2\left|0 - \frac{3}{6}\right| = 8.$$

Therefore, $E(H) = E(H_3)$.

**Theorem 2** *Let $G$ be the simple graph of order $n$ with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and $G^l$ be the graph obtained from $G$ by adding a loop on each vertex of $G$. Let $p$ and $q$ be non-negative integer and $p + q = m$ then $E((pG \cup qG^l)_{qn}) = mE(G)$, if $|\lambda_i| \geq max\left(\frac{p}{m}, \frac{q}{m}\right)$, for each $i = 1, 2, \cdots, n$.*

***Proof***: Let $H_{qn} = pG \cup qG^l$. The graph $H_{qn}$ contains $mn$ vertices and $qn$ loops. The adjacency matrix of $H_{qn}$ is given by:

$$A(H_{qn}) = \begin{bmatrix} A(G) & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & A(G) & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & A(G) & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & A(G)+I_n & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & A(G)+I_n & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & A(G)+I_n \end{bmatrix}$$

The characteristic polynomial of above matrix is given by:

$$\phi(H_{qn}:x) = \begin{vmatrix} xI - A(G) & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & 0 \\ 0 & \cdots & xI - A(G) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & xI-(A(G)+I_n) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & xI-(A(G)+I_n) \end{vmatrix}$$

It follows that if $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of $A$ then,

$$\phi(H_{qn}:x) = \prod_{i=1}^{n}(x-\lambda_i)^p(x-(\lambda_i+1))^q$$

The roots of above characteristic polynomial are:

$$x = \lambda_i(p - times),\ x = \lambda_i + 1(q - times)$$

, for each $i = 1, 2, \cdots, n$
Here,

$$\begin{aligned} E(H_{qn}) &= \sum_{i=1}^{n}\left(p\left|\lambda_i - \frac{qn}{mn}\right| + q\left|\lambda_i + 1 - \frac{qn}{mn}\right|\right) \\ &= \sum_{i=1}^{n}\left(p\left|\lambda_i - \frac{q}{m}\right| + q\left|\lambda_i + \frac{m-q}{m}\right|\right) \\ &= \sum_{i=1}^{n}\left(p\left|\lambda_i - \frac{q}{m}\right| + q\left|\lambda_i + \frac{p}{m}\right|\right) \end{aligned} \tag{2}$$

**Case – i** : Either $p > q$ **or** $p < q$
$\Rightarrow max\left(\frac{p}{m}, \frac{q}{m}\right) = \frac{p}{m}$ **or** $max\left(\frac{p}{m}, \frac{q}{m}\right) = \frac{q}{m}$

If $max\left(\frac{p}{m},\frac{q}{m}\right)=\frac{p}{m}$ then we suppose $|\lambda_i|\geq\frac{p}{m}>\frac{q}{m}$ **and**
if $max\left(\frac{p}{m},\frac{q}{m}\right)=\frac{q}{m}$ then we suppose $|\lambda_i|\geq\frac{q}{m}>\frac{p}{m}$, for all $1\leq i\leq n$.
Therefore,

$$\left|\lambda_i-\frac{q}{m}\right|=\begin{cases}|\lambda_i|-\frac{q}{m}, & \text{if } \lambda_i\geq 0\\ |\lambda_i|+\frac{q}{m}, & \text{if } \lambda_i<0\end{cases}$$

and

$$\left|\lambda_i+\frac{p}{m}\right|=\begin{cases}|\lambda_i|+\frac{p}{m}, & \text{if } \lambda_i\geq 0\\ |\lambda_i|-\frac{p}{m}, & \text{if } \lambda_i<0\end{cases}$$

Therefore, from equation 2

$$\begin{aligned}
E(H_{qn}) &= \sum_{i=1}^{n}\left(p\left|\lambda_i-\frac{q}{m}\right|+q\left|\lambda_i+\frac{p}{m}\right|\right)\\
&= \sum_{\lambda_i\geq 0}\left(p\left|\lambda_i-\frac{q}{m}\right|+q\left|\lambda_i+\frac{p}{m}\right|\right)+\sum_{\lambda_i<0}\left(p\left|\lambda_i-\frac{q}{m}\right|+q\left|\lambda_i+\frac{p}{m}\right|\right)\\
&= \sum_{\lambda_i\geq 0}\left(p|\lambda_i|-\frac{pq}{m}+q|\lambda_i|+\frac{pq}{m}\right)+\sum_{\lambda_i<0}\left(p|\lambda_i|+\frac{pq}{m}+q|\lambda_i|-\frac{pq}{m}\right)\\
&= p\left[\sum_{\lambda_i\geq 0}\left(|\lambda_i|-\frac{p}{m}+|\lambda_i|+\frac{p}{m}\right)+\sum_{\lambda_i<0}\left(|\lambda_i|+\frac{p}{m}+|\lambda_i|-\frac{p}{m}\right)\right]\\
&= p\left(\sum_{\lambda_i\geq 0}2|\lambda_i|+\sum_{\lambda_i<0}2|\lambda_i|\right)\\
&= 2pE(G)\\
&= (p+q)E(G)\\
&= mE(G)
\end{aligned}$$

**Case – ii** If $p=q$ then we assume $|\lambda_i|\geq\frac{p}{m}$, for all $1\leq i\leq n$ then

$$\left|\lambda_i-\frac{p}{m}\right|=\begin{cases}|\lambda_i|-\frac{p}{m}, & \text{if } \lambda_i\geq 0\\ |\lambda_i|+\frac{p}{m}, & \text{if } \lambda_i<0\end{cases}$$

and

$$\left|\lambda_i + \frac{p}{m}\right| = \begin{cases} |\lambda_i| + \frac{p}{m}, & \text{if } \lambda_i \geq 0 \\ |\lambda_i| - \frac{p}{m}, & \text{if } \lambda_i < 0 \end{cases}$$

Therefore, from equation 2

$$\begin{aligned} E(H_{qn}) &= \sum_{i=1}^{n} \left( p\left|\lambda_i - \frac{q}{m}\right| + p\left|\lambda_i + \frac{p}{m}\right| \right) \\ &= p\left[ \sum_{\lambda_i \geq 0} \left( \left|\lambda_i - \frac{p}{m}\right| + \left|\lambda_i + \frac{p}{m}\right| \right) + \sum_{\lambda_i < 0} \left( \left|\lambda_i - \frac{p}{m}\right| + \left|\lambda_i + \frac{p}{m}\right| \right) \right] \\ &= p\left[ \sum_{\lambda_i \geq 0} \left( |\lambda_i| - \frac{p}{m} + |\lambda_i| + \frac{p}{m} \right) + \sum_{\lambda_i < 0} \left( |\lambda_i| + \frac{p}{m} + |\lambda_i| - \frac{p}{m} \right) \right] \\ &= p\left( \sum_{\lambda_i \geq 0} 2|\lambda_i| + \sum_{\lambda_i < 0} 2|\lambda_i| \right) \\ &= 2pE(G) \\ &= (p+q)E(G) \\ &= mE(G) \end{aligned}$$

# Declarations

We here by declare that the research paper entitled “Some New Results on Energy of Graphs with Self Loops”submitted by us to Journal of Mathematical Chemistry. We further declare that the work done in this paper has not been submitted anywhere.

## Ethics approval

This declaration is “not applicable”.

## Competing interests

I declare that the authors have no competing interest or other interests that might be received to influence the results and/or discussion reported in this paper.

## Authors’ contributions

We jointly obtained some new results on energy of graphs with self-loops with equal contribution.

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This declaration is "not applicable".

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