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Analytical Determination of Deflection of Stepped Cantilever Rectangular Beam under Uniform load

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Abstract
Deflection of a beam is not only an interest for the branch of civil engineering but it has extended to many other engineering branches. Prediction of deflection for non-prismatic section with different loading conditions is a core area for many researchers. A simple analytical method to determine deflection of a non-prismatic beam (stepped beam) under uniform loading condition along with validation by FEA results are discussed in this research paper. The method provides accurate result for rectangular section. However, it can be applied for other sections as well.

Keywords: *Stepped beam, cantilever beam, non-prismatic beam, rectangular section*

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INTRODUCTION

A beam is a structure whose axial extension is predominant when compared to any other dimension orthogonal to it [1]. Prismatic beams have constant moment of inertia, whereas non-prismatic beams have variable moment of inertia [2]. Stepped beam is basically a non-prismatic cantilever beam as shown in Figure 1. In case of significant variation of the stress resultants along the length of member, economy can be achieved more efficiently by varying the cross-sectional area of the member, keeping in view the extreme values of the stress resultants in the middle and end sections [3]. This paper discusses deflection of stepped cantilever beam with uniform loading on one section.

LITERATURE

Non-prismatic beam has been an area of interest for many researchers. Jong has developed a method named method of model formulation [4]. This method is advantageous in a way; it doesn't put the need of many simultaneous equations especially when there are complicated situations. But solution of stepped beam is difficult with this method. Raymond investigated the slopes and deflections of a beam with two steps (i.e. a beam with a change of cross section) by use of a global formula [5].

Giuseppe *et al.* have discussed simple compatibility, equilibrium, and constitutive equations for a non-prismatic planar beam which is as per Timoshenko kinematics [6]. The method proposed can be used for wide verities of conditions i.e. loading and geometric conditions. The present paper finds deflection of a stepped cantilever beam with two sections. The approach presented here is very simple in comparison to the other methods presented by various researchers. All the above mentioned methods propose the use of many simultaneous equations. The present volume of this research paper presents a global formula instead of using many simultaneous equations.

DEFLECTION OF BEAM

As shown in Figure 1, two beams with different cross sectional area are taken into consideration. Length of the beam for smaller section beam is L_1 and for longer section beam is L_2. Let us assume that a force N acts entirely on the bigger section beam along its length. Two random sections are selected at a distance x_1 and x_2 from the fixed end.

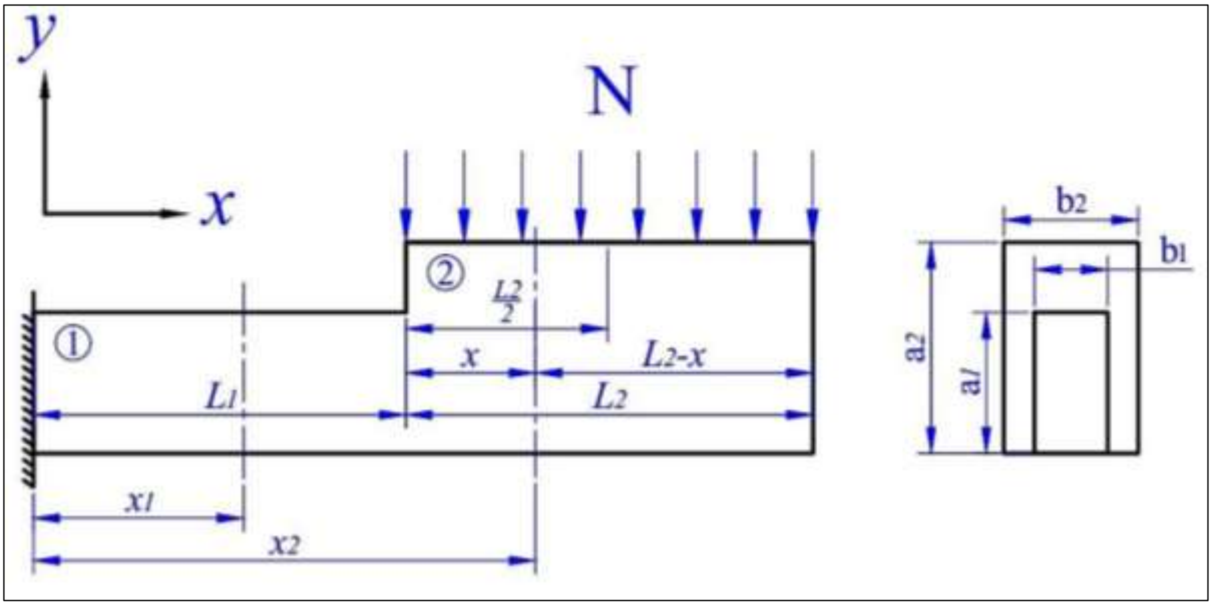

Fig. 1: *Stepped Rectangular Beam.*

Let us consider a beam-1 having length x_1, width b_1 and height a_1 and beam-2 having length x_2, width b_2 and height a_2.
Deflection of cantilever beam-1 can be calculated using Euler's equation as follows:

$$\frac{d^2y}{dx^2} = -\frac{M}{IE} \quad (1)$$

Where,
$y(x)$= deflection function,
M= Bending moment about C.G. axis at any distance x due to force F_x,
I= Area moment of inertia of cross section about C.G.,
E= Modulus of elasticity for isotropic material.
Force acting on section at distance x is:
$F_x = N$

Perpendicular distance from normal force F_x for any section at distance x_1 from the Figure 1 is,

$$d_x = \left(L_1 + \frac{L_2}{2} - x_1\right). \quad (2)$$

M_x= Force x Distance

$$M_x = N\left(L_1 + \frac{L_2}{2} - x_1\right)$$

$$M_x = N\left(L_1 + \frac{L_2}{2} - x_1\right) \quad (3)$$

Area moment of inertia for section I_x,

$$I_1(\text{C.G.}) = \frac{b_1 a_1^3}{12} \quad (4)$$

Solving Eq. (1) by placing values of M_x and I_1

$$\frac{d^2y}{dx^2} = \frac{N\left(L_1 + \frac{L_2}{2} - x_1\right)}{I_1 E} \quad (5)$$

$$y'(x) = \frac{-N\left(L_1 + \frac{L_2}{2} - x_1\right)^2}{2I_1E} + c_1 \qquad \text{where } c_1 \text{ is a constant} \quad (6)$$

$$y(x) = \frac{N\left(L_1 + \frac{L_2}{2} - x_1\right)^3}{6I_1E} + c_1x_1 + c_2 \text{ where } c_2 \text{ is a constant} \quad (7)$$

Boundary condition for cantilever beam with point loading are:

$y(0)=0$ and $y'(0)=0$

$$0=\frac{-N\left(L_1+\frac{L_2}{2}-0\right)^2}{2I_1E}+c_1$$

$$c_1=\frac{N\left(L_1+\frac{L_2}{2}\right)^2}{2I_1E} \tag{8}$$

$$0=\frac{N\left(L_1+\frac{L_2}{2}-0\right)^3}{6I_1E}+c_1(0)+c_2$$

$$c_2=-\frac{N\left(L_1+\frac{L_2}{2}\right)^3}{6I_1E} \tag{9}$$

Substituting values of c_1 and c_2 from Eqs. (8) and (9) in Eq. (7)

$$y(x)=\frac{N\left(L_1+\frac{L_2}{2}-x_1\right)^3}{6I_1E}+\frac{N\left(L_1+\frac{L_2}{2}\right)^2x_1}{2I_1E}-\frac{N\left(L_1+\frac{L_2}{2}\right)^3}{6I_1E} \tag{10}$$

$$y'(x)=\frac{-N\left(L_1+\frac{L_2}{2}-x_1\right)^2}{2I_1E}+\frac{N\left(L_1+\frac{L_2}{2}\right)^2}{2I_1E} \tag{11}$$

$$y(L_1)=\frac{N\left(L_1+\frac{L_2}{2}-L_1\right)^3}{6I_1E}+\frac{N\left(L_1+\frac{L_2}{2}\right)^2L_1}{2I_1E}+\frac{N\left(L_1+\frac{L_2}{2}\right)^3}{6I_1E}$$

$$y(L_1)=\frac{N\left(\frac{L_2}{2}\right)^3}{6I_1E}+\frac{N\left(L_1+\frac{L_2}{2}\right)^2L_1}{2I_1E}+\frac{N\left(L_1+\frac{L_2}{2}\right)^3}{6I_1E} \tag{11.1}$$

And from Eq. (11),

$$y'(L_1)=\frac{-N\left(L_1+\frac{L_2}{2}-L_1\right)^2}{2I_1E}+\frac{N\left(L_1+\frac{L_2}{2}\right)^2}{2I_1E}$$

$$y'(L_1)=\frac{-N\left(\frac{L_2}{2}\right)^2}{2I_1E}+\frac{N\left(L_1+\frac{L_2}{2}\right)^2}{2I_1E} \tag{11.2}$$

Similarly, deflection of cantilever beam-2 can be calculated using Euler's equation as follows:

$$\frac{d^2y}{dx^2}=-\frac{M_x}{I_xE}\quad. \tag{12}$$

Where,

$y(x)$, M_x, I_x, E are corresponding values for beam-2.

Force acting on section at distance x is:

$$F_x=\frac{N(L_2-x)}{L_2}. \tag{13}$$

Perpendicular distance from normal force F_x for any section at distance x from the Figure 1 is,

$$d_x=\frac{L_2-x}{2}. \tag{14}$$

M_x = Force x Distance

$$M_x=\frac{N(L-x)}{L}\cdot\frac{L-x}{2}$$

$$M_x=\frac{N(L-x)^2}{2L} \tag{15}$$

Area moment of inertia for section I_x,

$$I_{2(C.G.)}=\frac{b_2a_2^3}{12}.$$

Solving Eq. (12) by placing values of M_x and I_z

$$\frac{d^2y}{dx^2}=\frac{N(L_2-x_2)^2}{2L_2I_2E} \tag{16}$$

$$y'(x)=\frac{-N(L_2-x_2)^3}{6L_2I_2E}+c_3 \text{ where } c_3 \text{ is a constant} \tag{17}$$

$$y(x)=\frac{N(L_2-x_2)^4}{24L_2I_2E}+c_3x_2+c_4 \text{.... where } c_4 \text{ is a constant} \tag{18}$$

Boundary condition for cantilever beam with point loading are $y(0) = 0$ and $y'(0) = 0$
This boundary condition is to be applied for beam-2. So, here let us find $y(L_1)$ and $y'(L_1)$. From Eq. (11.2),

$$y'(L_1)=\frac{-N\left(\frac{L_2}{2}\right)^2}{2I_1E}+\frac{N\left(L_1+\frac{L_2}{2}\right)^2}{2I_1E} \tag{19}$$

Substituting the value of c_1 from Eq. (8) to Eq. (19)
Now, from Eq. (11.1), $y'(L_1)=y'(x)$ in Eq. (17)

$$\therefore \frac{-N\left(\frac{L_2}{2}\right)^2}{2I_1E}+\frac{N\left(L_1+\frac{L_2}{2}\right)^2}{2I_1E}=\frac{-N(L_2-x_2)^3}{6L_2I_2E}+c_3$$

$$c_3=\frac{-N\left(\frac{L_2}{2}\right)^2}{2I_1E}+\frac{N\left(L_1+\frac{L_2}{2}\right)^2}{2I_1E}+\frac{N(L_2-L_1)^3}{6L_2I_2E} \tag{20}$$

From Eq. (11.1), putting $y(x)=y(L_1)$ in Eq. (18)

$$\frac{N\left(\frac{L_2}{2}\right)^3}{6I_1E}+\frac{N\left(L_1+\frac{L_2}{2}\right)^2L_1}{2I_1E}-\frac{N\left(L_1+\frac{L_2}{2}\right)^3}{6I_1E}=\frac{N(L_2-x_2)^4}{24L_2I_2E}+\left(\frac{-N\left(\frac{L_2}{2}\right)^2}{2I_1E}+\frac{N\left(L_1+\frac{L_2}{2}\right)^2}{2I_1E}+\frac{N(L_2-L_1)^3}{6L_2I_2E}\right)L_1+c_4$$

$$\therefore c_4=\frac{N\left(\frac{L_2}{2}\right)^3}{6I_1E}+\frac{N\left(L_1+\frac{L_2}{2}\right)^2L_1}{2I_1E}-\frac{N\left(L_1+\frac{L_2}{2}\right)^3}{6I_1E}-\frac{N(L_2-x_2)^4}{24L_2I_2E}-\left(\frac{-N\left(\frac{L_2}{2}\right)^2}{2I_1E}+\frac{N\left(L_1+\frac{L_2}{2}\right)^2}{2I_1E}+\frac{N(L_2-L_1)^3}{6L_2I_2E}\right)L_1 \tag{21}$$

Inserting the values of $c3$ and $c4$ in Eq. (18),

$$Y_2(x_2)=\frac{N(L_2-x_2)^4}{24L_2I_2E}+\frac{-N\left(\frac{L_2}{2}\right)^2}{2I_1E}+\frac{N\left(L_1+\frac{L_2}{2}\right)^2}{2I_1E}+\frac{N(L_2-L_1)^3}{6L_2I_2E}x_2+\frac{N\left(\frac{L_2}{2}\right)^3}{6I_1E}+\frac{N\left(L_1+\frac{L_2}{2}\right)^2L_1}{2I_1E}-\frac{N\left(L_1+\frac{L_2}{2}\right)^3}{6I_1E}-\frac{N(L_2-x_2)^4}{24L_2I_2E}-\left(\frac{-N\left(\frac{L_2}{2}\right)^2}{2I_1E}+\frac{N\left(L_1+\frac{L_2}{2}\right)^2}{2I_1E}+\frac{N(L_2-L_1)^3}{6L_2I_2E}\right)L_1 \tag{22}$$

Eq. (22) shows deflection of a stepped beam with rectangular sections at length x_2 where $x_2>x_1$.

SOLUTION USING EXCEL

To validate above equation let us assume the dimensions as per Table 1. See Figure 1 for parameters.

Table 1: *Assumed Parameters.*

Parameters	Value
L1	500 mm
a1	50 mm
b1	50 mm
L2	500 mm
a2	100 mm
b2	50 mm
N	1000 N
E	200000 MPa

Solutions of the equations are done using Microsoft Excel, screen shot of which is shown in Figure 2.

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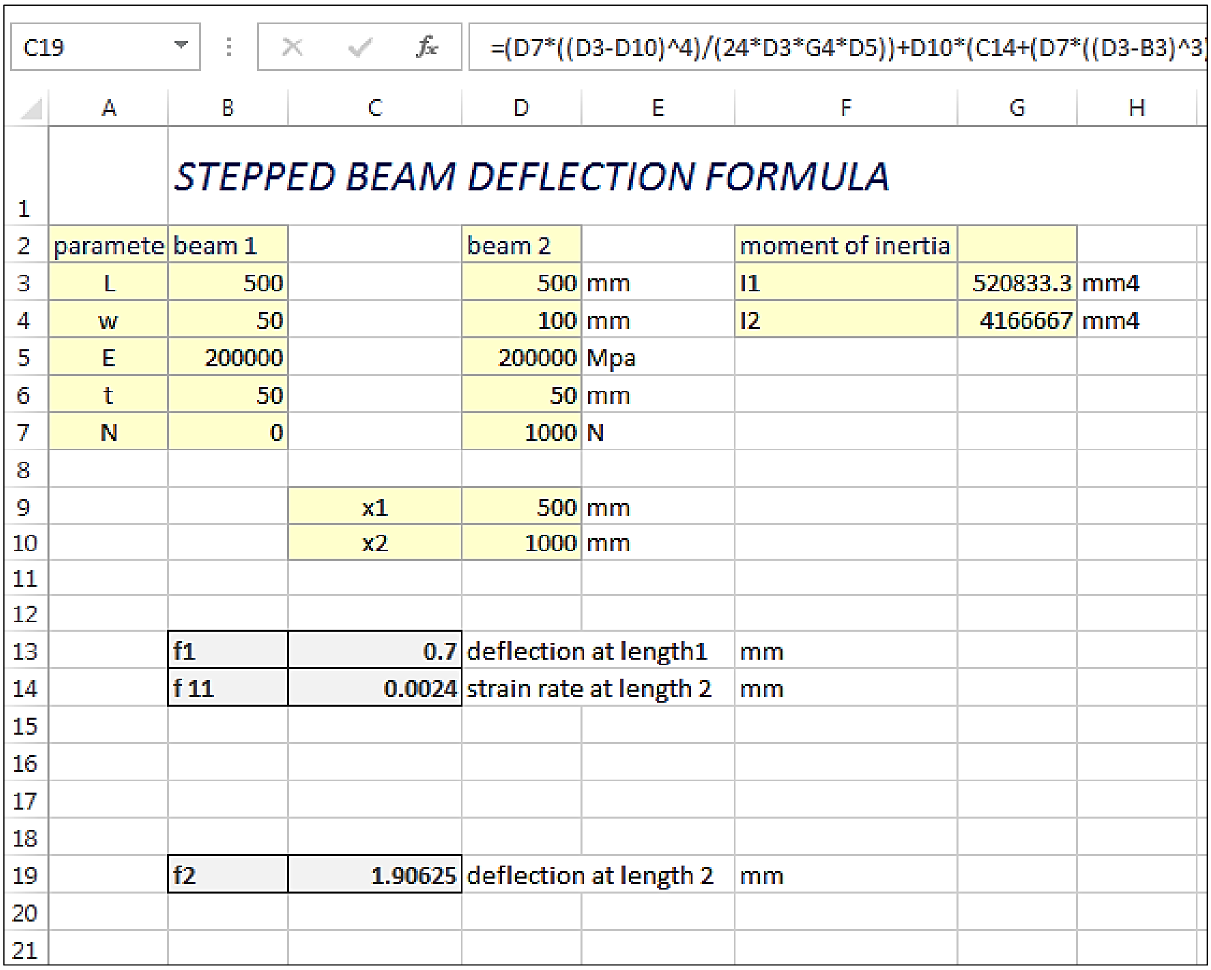
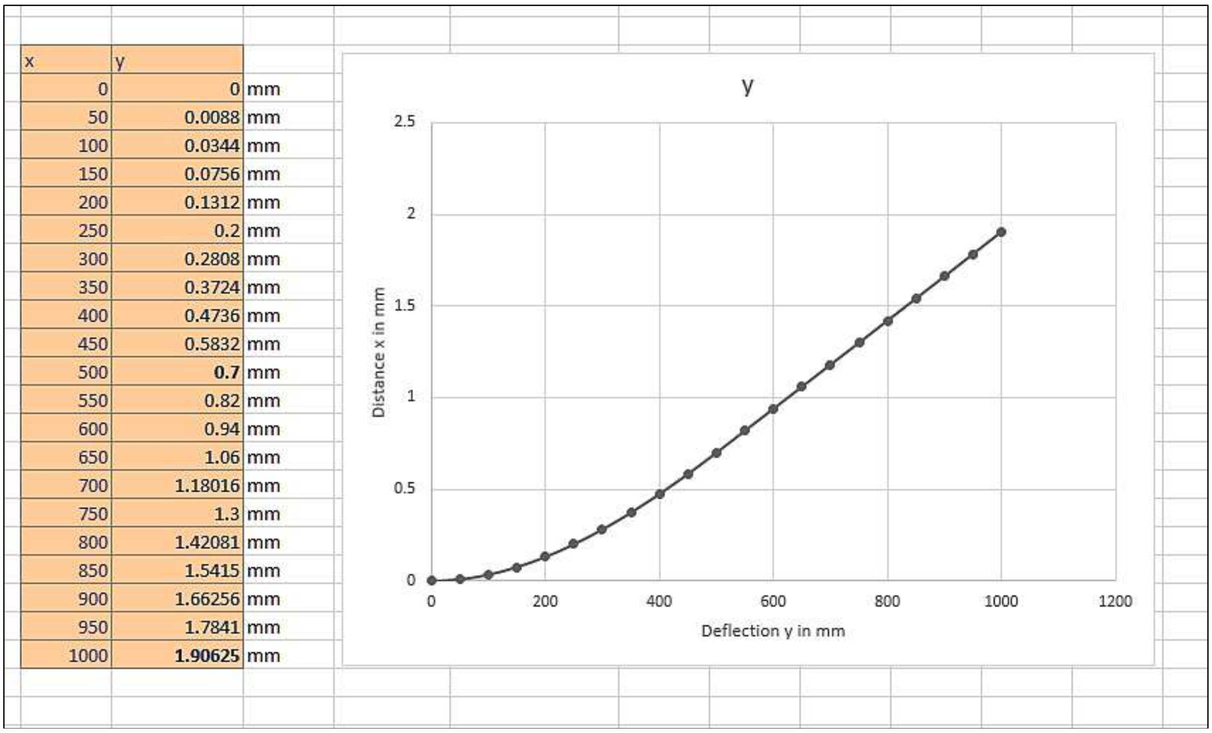


C19 =(D7*((D3-D10)^4)/(24*D3*G4*D5))+D10*(C14+(D7*((D3-B3)^3)

	A	B	C	D	E	F	G	H
1		STEPPED BEAM DEFLECTION FORMULA						
2	paramete	beam 1		beam 2		moment of inertia		
3	L	500		500	mm	I1	520833.3	mm4
4	w	50		100	mm	I2	4166667	mm4
5	E	200000		200000	Mpa			
6	t	50		50	mm			
7	N	0		1000	N			
8								
9			x1	500	mm			
10			x2	1000	mm			
11								
12								
13		f1	0.7	deflection at length1		mm		
14		f 11	0.0024	strain rate at length 2		mm		
15								
16								
17								
18								
19		f2	1.90625	deflection at length 2		mm		
20								
21								

Fig. 2: *Screen Shot of Excel Analysis.*



x	y	
0	0	mm
50	0.0088	mm
100	0.0344	mm
150	0.0756	mm
200	0.1312	mm
250	0.2	mm
300	0.2808	mm
350	0.3724	mm
400	0.4736	mm
450	0.5832	mm
500	0.7	mm
550	0.82	mm
600	0.94	mm
650	1.06	mm
700	1.18016	mm
750	1.3	mm
800	1.42081	mm
850	1.5415	mm
900	1.66256	mm
950	1.7841	mm
1000	1.90625	mm

Fig. 3: *Deflection Graph.*

Figure 3 shows the deflection value of the complete beam in the interval of 50 mm. It is to be noted that negative deflection is converted into positive deflection for simplicity. Figure 3 also shows deflection graph. Maximum deflection is 1.906 mm for the length of 1 m.

FINITE ELEMENT ANALYSIS

Finite element analysis is carried out to compare analytical result. FEA is done using Creo3.0 as shown in Figure 4. The element size is restricted to maximum of 20 mm. 7202 elements are created. Maximum deflection found is 1.939 mm.

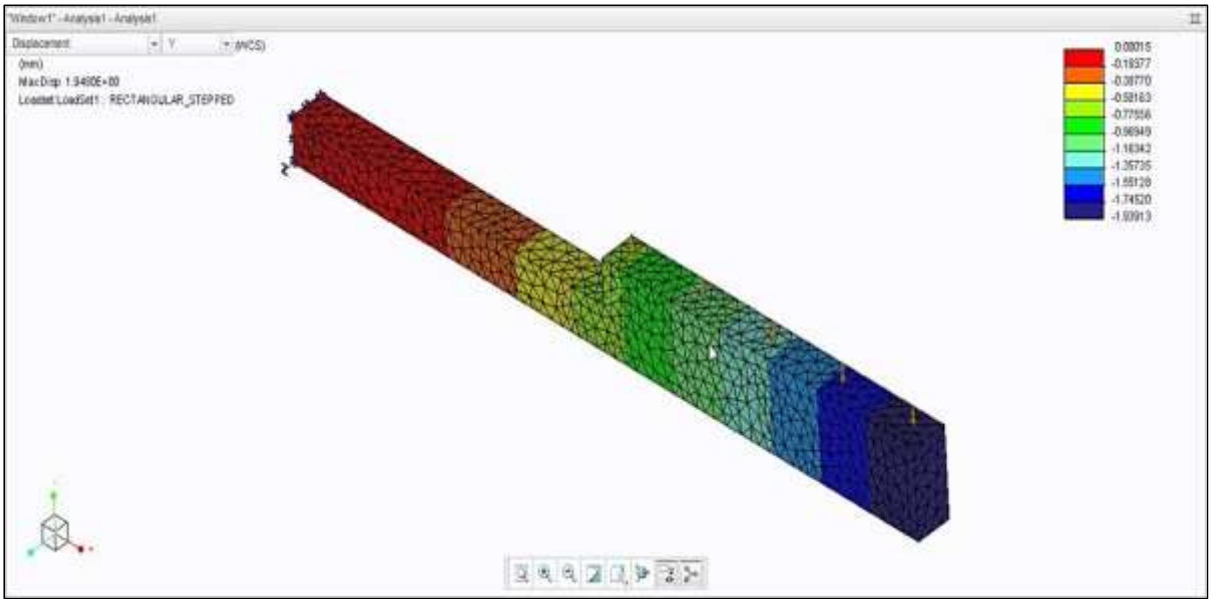


Fig. 4: *FEA for Stepped Rectangular Beam.*

Figure 5 shows deflection curve generated from the Creo analysis, which shows negative deflection.

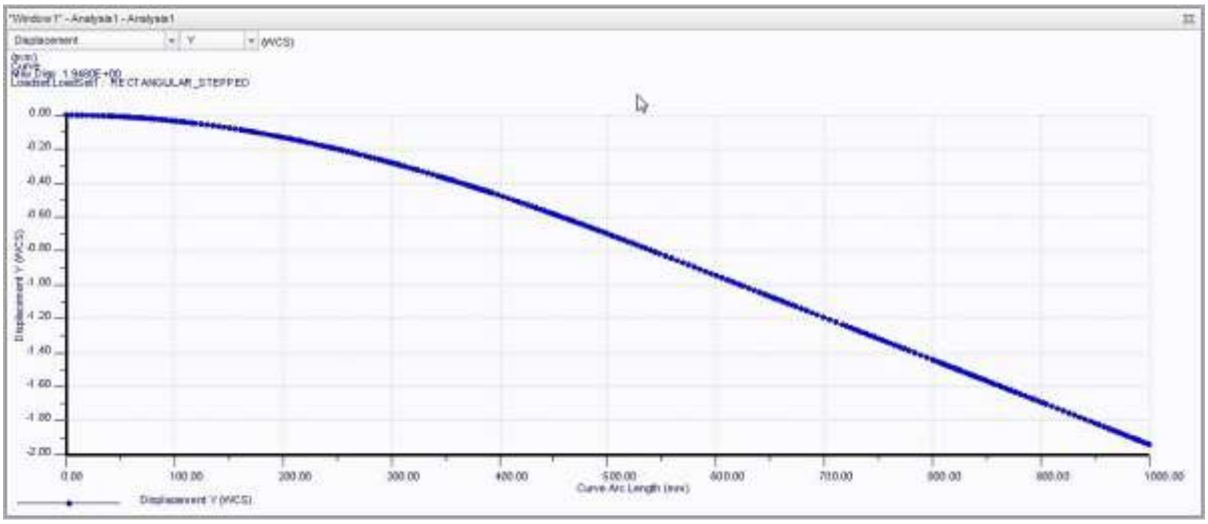


Fig. 5: *Deflection Curve from Creo.*

COMPARISON

After exporting above curve with data file into Excel, and matching with the analytical calculations, following result is found, as shown in Figure 6. It shows the deflection curves determined from analytical method and Creo simulation.

Table 2 shows comparison of analytical and FEA results. The difference of deflection between analytical and FEA result is 0.03287 mm and percentage difference is 1.7%. This difference is for the length of 1 m.

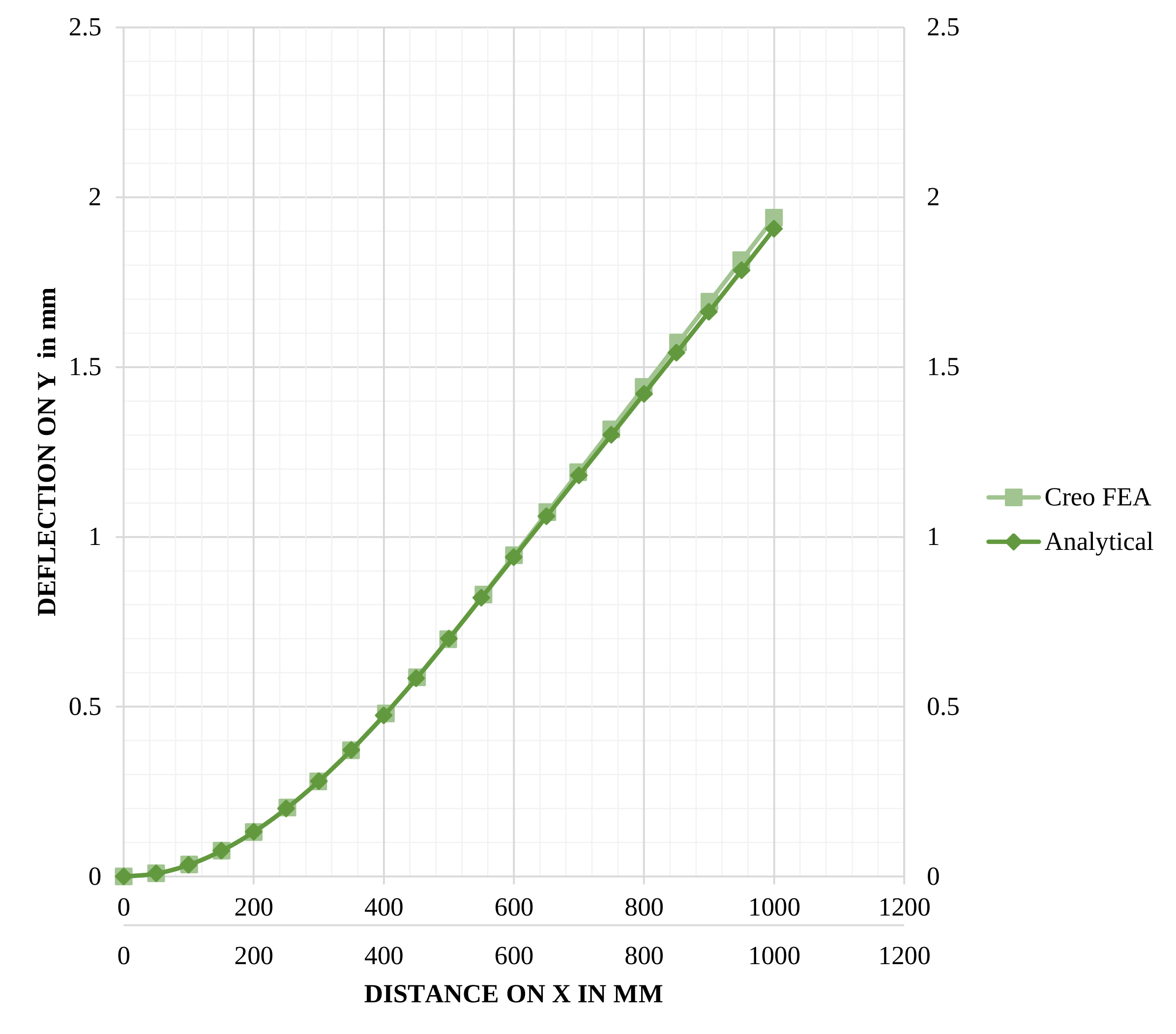


Fig. 6: *Comparison of FEA and Analytical Result.*

Table 2: *Result Comparison.*

Sr. No.	Type of Analysis	Maximum Deflection (mm)	Dimensional Difference (mm)	Percentage Difference
1.	Analytical	1.90625	-	-
2.	FEA	1.93912	0.03287	1.7

CONCLUSION

The method suggested here gives very accurate result. It uses double integration method and is quite simple in comparison to other methods. The results obtained are accurate for rectangular section. But it can be conveniently applied to other sections as well where moment of inertia is known. Whole method can also be converted into simple excel programming.

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