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Characteristic polynomials of some algebraic graphs

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Abstract
The zero divisor graph $\Gamma(R)$ of a commutative ring R is a graph whose vertices are non-zero zero divisors of R and two vertices are adjacent if their product is zero. The characteristic polynomial of matrix M is defined as $|\lambda I - M|$ and roots of the characteristic polynomial are known as eigenvalues of M. We investigate eigenvalues and characteristic polynomials for some zero divisor graphs.

Keywords
Zero-divisor Graph, Adjacency Matrix, Characteristic Polynomial, Eigenvalue, Energy.

AMS Subject Classification
05C25, 05C50.

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Contents

1. Introduction

The concept of zero divisor graph of commutative ring R was introduced by Beck [5] in 1988. In last two decades the zero divisor graph is extensively studied by many researchers [2–4, 8, 9]. For any matrix M, the characteristic polynomial is defined as $|\lambda I - M|$ and roots of the characteristic polynomial are called eigenvalues of M. The concept of energy of graph was introduced by Gutman [6] in 1978. The study of energy of zero divisor graph was first initiated by Ahmadi and Nezhad [1] for the ring $\mathbb{Z}_n$ for $n = p^2$ and $n = pq$, where p and q are distinct primes. The adjacency matrix and eigenvalues of the zero divisor graph $\Gamma(\mathbb{Z}_n)$ for $n = p^3$ and $n = p^2q$ was studied by Reddy *et al.* [10].

In this paper, we study the energy and characteristic polynomial of zero divisor graph $\Gamma(\mathbb{Z}_n)$ for $n = p^4$ and zero divisor graphs obtained from direct product of rings. Throughout this paper we consider the commutative ring R with unity. If R is a ring then $Z(R)$ and $Z^*(R)$ denote the set of zero divisors and set of non-zero zero divisors of the ring R respectively. The zero divisor graph of a ring R, denoted as $\Gamma(R)$, is a graph whose vertices are the non-zero zero divisors and two vertices are adjacent if and only if their product is zero. We use $M(\Gamma(R))$ to denote the adjacency matrix of $\Gamma(R)$ and $E(\Gamma(R))$ for the energy, defined as sum of modulus of eigenvalues of graph, of $\Gamma(R)$ and the matrix with all entries 1 will be denoted as J.

2. Main Results

Proposition 2.1. *[7] Let* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *be any matrix. Then* $|M| = |A||D - CA^{-1}B|$

Theorem 2.2. *Let $n = p^4$ with p any prime. If λ is any nonzero eigenvalue of $\Gamma(\mathbb{Z}_n)$ then,*
$\lambda^3 - \lambda^2\left(p^2-1\right) - \lambda p^2 (p-1)^2 + p^3 (p-1)^3 = 0$

Proof. Let $n = p^4$. Then the set of non-zero zero divisors of $\mathbb{Z}_n$ is $Z^*(\mathbb{Z}_n) = \{p, 2p, 3p, ..., (p^3-1)p\}$. We partition the set $Z^*(\mathbb{Z}_n)$ as $Z^*(\mathbb{Z}_n) = A \cup B \cup C$, where $A = \{k_1 p \mid k_1 = 1,2,3,...,p^3-1 \text{ and } p \nmid k_1\}$, $B = \{k_2 p^2 \mid k_2 = 1,2,3,...,p^2-1 \text{ and } p \nmid k_2\}$ and $C = \{k_3 p^3 \mid k_3 = 1,2,3,...,p-1\}$. Then $|A| = p^3 - p^2$, $|B| = p^2 - p$, and $|C| = p-1$. Since the elements of A and B are not adjacent, we get the zero matrices of order $p^3 - p^2$, $\left(p^3-p^2\right) \times \left(p^2-p\right)$ and $\left(p^2-p\right) \times \left(p^3-p^2\right)$. As the elements of A and C are adjacent implies we get a matrix of ones of order $(p^3-p^2) \times (p-1)$. Similarly we get the matrices corresponding to B & B, B & C,

C & A,C & C. Hence we get the adjacency matrix by considering the elements of A first, then B and then C as

$$M\left(\Gamma(\mathbb{Z}_{p^4})\right) = \begin{bmatrix} O & O & J \\ O & J & J \\ J & J & J \end{bmatrix}_{(p^3-1)\times(p^3-1)}$$

where O is the zero matrix and J is the matrix of ones. Let λ be any eigenvalue of $M\left(\Gamma(\mathbb{Z}_{p^4})\right)$. Then

$$|\lambda I - M| = \begin{vmatrix} [\lambda I - O] & O & J \\ O & [\lambda I - J] & J \\ J & J & [\lambda I - J] \end{vmatrix} = 0$$

Let $T_1 = \begin{bmatrix} [\lambda I - O] & O \\ O & [\lambda I - J] \end{bmatrix}_{(p^3-p)\times(p^3-p)}$

$T_2 = [-J]_{(p^3-p)\times(p-1)}$, $T_3 = [-J]_{(p^-1)\times(p^3-p)}$ and $T_4 = [\lambda I - J]_{(p-1)\times(p-1)}$

Then by Proposition 2.1, $|M-\lambda I| = |T_1||T_4 - T_3 T_1^{-1} T_2| = 0$.

Now by straight forward calculation we get $|T_1| = \lambda^{p^3-p-1}\left(\lambda - \left(p^2-p\right)\right)$,

$T_1^{-1} = \frac{1}{\lambda}\begin{bmatrix} I & O \\ O & \frac{1}{\lambda - p^2 + p}((\lambda - p^2 + p)I + J) \end{bmatrix}$

and $T_3 T_1^{-1} T_2 = \left(\frac{(p^3-p)\lambda - p^3(p-1)^2}{\lambda\left(\lambda - (p^2-p)\right)}\right) J_{(p-1)\times(p-1)}$

Therefore $|T_4 - T_3 T_1^{-1} T_2| = \lambda^{p-2}\left(\frac{\lambda^3 - \lambda^2(p^2-1) - \lambda p^2(p-1)^2 + p^3(p-1)^3}{\lambda\left(\lambda - (p^2-p)\right)}\right)$

Hence $|M-\lambda I| = |T_1||T_4 - T_3 T_1^{-1} T_2| = \left(\lambda^{p^3-p-1}\left(\lambda - (p^2-p)\right)\right)\left(\lambda^{p-2}\left(\frac{\lambda^3 - \lambda^2(p^2-1) - \lambda p^2(p-1)^2 + p^3(p-1)^3}{\lambda\left(\lambda - (p^2-p)\right)}\right)\right) = 0$

Thus characteristic polynomial is $\lambda^{p^3-4}\left(\lambda^3 - \lambda^2\left(p^2-1\right) - \lambda p^2\left(p-1\right)^2 + p^3\left(p-1\right)^3\right) = 0$

Hence we get $\lambda^3 - \lambda^2\left(p^2-1\right) - \lambda p^2\left(p-1\right)^2 + p^3\left(p-1\right)^3 = 0$ for any non zero eigenvalue λ of $M\left(\Gamma(\mathbb{Z}_{p^4})\right)$. ☐

Theorem 2.3. *Let $\mathbb{Z}_p \times \mathbb{Z}_p$ be a ring with p be any prime then $E(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)) = 2(p-1)$.*

Proof. Being $\mathbb{Z}_p \times \mathbb{Z}_p$ ring, $\mathbb{Z}_p$ has no non-zero divisor implies $Z^*(\mathbb{Z}_p \times \mathbb{Z}_p) = A \cup B$, where $A = \{(0,kp)\,|k = 1,2,3,...,p-1\}$ and $B = \{(kp,0)\,|k = 1,2,3,4,...,p-1\}$, moreover $|Z^*(\mathbb{Z}_p \times \mathbb{Z}_p)| = 2(p-1)$, $|A| = p-1$ and $|B| = p-1$. Since every element of A is adjacent with every element of B, we get the matrix M_1 with all entries ones of order $(p-1)\times(p-1)$. Hence we get the adjacency matrix by considering the elements of A first and then B as

$$M(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

then by simple calculation we get $|\lambda I - M(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p))| =$

$$\begin{vmatrix} \lambda & 0 & 0 & \cdots & 0 & -1 & -1 & -1 & \cdots & -1 \\ 0 & \lambda & 0 & \cdots & 0 & -1 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & -1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & -1 & \cdots & -1 & \lambda & 0 & 0 & \cdots & 0 \\ -1 & -1 & -1 & \cdots & -1 & 0 & \lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix} = 0$$

$|\lambda I - M(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p))| = \lambda^{2p-4}\left(\lambda^2 - (p-1)^2\right) = 0$

$\therefore \lambda = 0, p-1, -(p-1)$. Now the energy of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)$ is given by

$$E\left(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)\right) = \sum_{i=1}^{2p-2} \lambda_i$$

$$E\left(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_p)\right) = 2(p-1)$$

☐

Theorem 2.4. *Let $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ be a ring with p be any prime. If λ is any non-zero eigenvalue of the adjacency matrix $M\left(\Gamma\left(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\right)\right)$ then λ satisfies the equation $\lambda^4 + \lambda^3(p-1) + \lambda^2\left(2(p-1)^3\right) + \lambda\left(p(p-1)^3\right) + p(p-1)^5 = 0$.*

Proof. Let $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$ be a ring. Note that $\mathbb{Z}_p$ has no non-zero zero divisor and in $\mathbb{Z}_{p^2}$ the non-zero zero divisors are multiples of p. Hence $Z^*\left(\mathbb{Z}_p \times \mathbb{Z}_{p^2}\right) = A_1 \cup A_2 \cup A_3 \cup A_4$ where $A_1 = \{(x_1,kp)\,|x_1 \in \mathbb{Z}_p^*\ \&\ k = 1,2,3,...,p-1\}$, $A_2 = \{(0,x_2)\,|x_2 \in \mathbb{Z}_{p^2}^*\}$, $A_3 = \{(x_1,0)\,|n_1 \in \mathbb{Z}_p^*\}$ and $A_4 = \{(0,kp)\,|k = 1,2,3,...,p-1\}$.

Then $|A_1| = (p-1)^2$, $|A_2| = (p^2-p)$, $|A_3| = (p-1)$, $|A_4| = (p-1)$. Since no element of A_1 is adjacent with element of A_1, A_2 and A_3, we get the zero matrices of order $(p-1)^2 \times (p-1)^2$, $(p-1)^2 \times (p^2-p)$ and $(p-1)^2 \times (p-1)$ respectively. Also we get zero matrices corresponding to A_2 and A_2;A_3 and A_3; and A_2 and A_4. And every element of A_1 is adjacent with every element of A_4 implies we get a matrix of order $(p-1)^2 \times (p-1)$ whose all entries are ones. Similarly we get matrices of ones corresponding to A_4 and A_4; and A_3 and A_4.

Hence we get the adjacency matrix by considering the elements of A_1 first, then A_2, then A_3 and then A_4 as

$$M(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})) = \begin{bmatrix} O & O & O & R_1 \\ O & O & R_2 & O \\ O & R_2^T & O & R_3 \\ R_1^T & O & R_3^T & R_4 \end{bmatrix}$$

where O is the zero matrix and R_i is the matrix of ones for $i = 1,2,3,4$.

Let λ be any non-zero eigenvalue of $M(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2}))$, then

$$|\lambda I - M(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2}))| = \begin{vmatrix} \lambda I & O & O & R_1 \\ O & \lambda I & R_2 & O \\ O & R_2^T & \lambda I & R_3 \\ R_1^T & O & R_3^T & \lambda I - R_4 \end{vmatrix} = 0$$

Let $T_1 = \lambda I$, $T_2 = \begin{bmatrix} 0 & -R_1 \\ -R_2 & 0 \end{bmatrix}$
$T_3 = \begin{bmatrix} 0 & -R_2^T \\ -R_1^T & 0 \end{bmatrix}$ and $T_4 = \begin{bmatrix} \lambda I & -R_3 \\ -R_3^T & \lambda I - R_4 \end{bmatrix}$

Then by Proposition 2.1 $|\lambda I - M(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2}))| = \begin{vmatrix} T_1 & T_2 \\ T_3 & T_4 \end{vmatrix}$
$= |T_1||T_4 - T_3 T_1^{-1} T_2| = 0$.
Since T_1 is a scalar matrix of order $(p-1)(2p-1)$, we get $|T_1| = \lambda^{(p-1)(2p-1)}$.
And $T_3 T_1^{-1} T_2 = \begin{bmatrix} \frac{p^2-p}{\lambda} J & O \\ O & \frac{(p-1)^2}{\lambda} J \end{bmatrix}$

So $|T_4 - T_3 T_1^{-1} T_2| = \begin{vmatrix} \lambda I - \frac{p^2-p}{\lambda} J & O \\ O & (\lambda - 1) I \frac{(p-1)^2}{\lambda} J \end{vmatrix}$
$= \lambda^{2p-6}(\lambda^4 + \lambda^3(p-1) + \lambda^2(2(p-1)^3) + \lambda(p(p-1)^3) + p(p-1)^5)$.
Therefore $|\lambda I - M(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2}))| = \lambda^{p(2p-1)-5}(\lambda^4 + \lambda^3(p-1) + \lambda^2(2(p-1)^3) + \lambda(p(p-1)^3) + p(p-1)^5) = 0$ which is the characteristic polynomial of $\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{p^2})$.
Since $\lambda \neq 0$, we get $\lambda^4 + \lambda^3(p-1) + \lambda^2(2(p-1)^3) + \lambda(p(p-1)^3) + p(p-1)^5) = 0$

□

Theorem 2.5. *Let $\mathbb{Z}_p \times \mathbb{Z}_{pq}$ be a ring with p, q be distinct primes. Then the spectra of zero divisor graph of $\mathbb{Z}_p \times \mathbb{Z}_{pq}$ is* $Spec(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{pq})) =$

$$\begin{pmatrix} 0 & \frac{1}{2}(p-1)(-1 \pm \sqrt{4q-3}) & \lambda_i \\ (p-1)(p+2q-3)+2p+q-9 & 1 & 1 \end{pmatrix}$$

,for $i = 1,2,3,4$, where λ_i is the solution of the equation $\lambda^4 - \lambda^3(p-1) - \lambda^2(2p(p-1)(q-1)) + \lambda(p-1)^3(q-1) + (p-1)^4(q-1)^2 = 0$

Proof. Let $\mathbb{Z}_p \times \mathbb{Z}_{pq}$ be a ring with p, q be distinct primes. Note that the set of zero divisors of $\mathbb{Z}_p \times \mathbb{Z}_{pq}$ is given by $Z^*(\mathbb{Z}_p \times \mathbb{Z}_{pq}) = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$, where $A_1 = \{(x_1, k_1 p) \,|x_1 \in \mathbb{Z}_p^* \,\&\, k_1 = 1,2,3,...,q-1\}$, $A_2 = \{(0, x_2) \,|x_2 \in \mathbb{Z}_{pq}^*$ *and x_2 is non zero divisor* $\}$, $A_3 = \{(x_1, k_2 q) \,|x_1 \in \mathbb{Z}_p^* \,\&\, k_2 = 1,2,3,...,p-1\}$, $A_4 = \{(0, k_2 q) \,|k_2 = 1,2,3,...,p-1\}$, $A_5 = \{(x_1, 0) \,|n_1 \in \mathbb{Z}_p^*\}$ and $A_6 = \{(0, k_1 p) \,|k_1 = 1,2,3,...,q-1\}$.
Then $|A_1| = (p-1)(q-1)$, $|A_2| = (p-1)(q-1)$, $|A_3| = (p-1)^2$, $|A_4| = (p-1)$, $|A_5| = (p-1)$, & $|A_6| = (q-1)$.
Since all the elements of A_1 is adjacent with all the elements of A_4, we get the matrix of ones. Similarly we get the matrices of ones corresponding to A_2 & A_5;A_3 & A_6;A_4 & A_5;A_4 & A_6; and A_5 & A_6. As no element of A_1 is adjacent with element of A_1, A_2, A_3,A_5, A_6, we get zero matrix. Similarly we get the zero matrices corresponding to rest of the pairs.
Hence we get adjacency matrix of zero divisor graph of $\mathbb{Z}_p \times \mathbb{Z}_{pq}$ by considering A_1 first, then A_2, then A_3, then A_4, then A_5 and then A_6 as $M(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{pq})) =$

$$\left[\begin{array}{c|c} [O]_{(p-1)(p+2q-3)\times(p-1)(p+2q-3)} & [S_1]_{(p-1)(p+2q-3)\times(2p+q-3)} \\ \hline [S_1^T]_{(2p+q-3)\times(p-1)(p+2q-3)} & [S_2]_{(2p+q-3)\times(2p+q-3)} \end{array}\right]$$

,where O is the zero matrix of order $(p-1)(p+2q-3) \times (p-1)(p+2q-3)$ and

$$S_1 = \begin{bmatrix} J & O & O \\ O & J & O \\ O & O & J \end{bmatrix} \quad S_2 = \begin{bmatrix} O & J & J \\ J & O & J \\ J & J & O \end{bmatrix}$$

Let λ be any non-zero eigenvalue of $M(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{pq}))$, then $|\lambda I - M(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{pq}))| = 0$

Now let $T_1 = \lambda I - O$, $T_2 = S_1$, $T_3 = S_1^T$ and $T_4 = \lambda I - S_2$.
Then by Proposition 2.1 $|\lambda I - M(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{pq}))| = \begin{vmatrix} T_1 & T_2 \\ T_3 & T_4 \end{vmatrix} =$
$|T_1||T_4 - T_3 T_1^{-1} T_2| = 0$.
Since T_1 is a scalar matrix of order $(p-1)(p+2q-3)$, we get $|T_1| = \lambda^{(p-1)(p+2q-3)}$. And

$$T_3 T_1^{-1} T_2 = \begin{bmatrix} \frac{(p-1)(q-1)}{\lambda} J & O & O \\ O & \frac{(p-1)(q-1)}{\lambda} J & O \\ O & O & \frac{(p-1)^2}{\lambda} J \end{bmatrix}$$

and $|T_4 - T_3 T_1^{-1} T_2| = \lambda^{2p+q-9}(\lambda^2 + \lambda(p-1) - (p-1)^2 (q-1))(\lambda^4 - \lambda^3(p-1) - \lambda^2(2p(p-1)(q-1)) + \lambda(p-1)^3 (q-1) + (p-1)^4(q-1)^2)$.
Therefore $|\lambda I - M(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_{pq}))| = \lambda^{(p-1)(p+2q-3)+2p+q-9} (\lambda^2 + \lambda(p-1) - (p-1)^2(q-1))(\lambda^4 - \lambda^3(p-1) - \lambda^2(2p (p-1)(q-1)) + \lambda(p-1)^3(q-1) + (p-1)^4(q-1)^2)$ which is characteristic polynomial. Hence the proof. □

Theorem 2.6. *Let $\mathbb{Z}_p \times \mathbb{Z}_{p^3}$ be a ring with p be any prime. If λ is any non-zero eigenvalue of the adjacency matrix $M\left(\Gamma\left(\mathbb{Z}_p \times \mathbb{Z}_{p^3}\right)\right)$ then λ satisfies the equation $\lambda^4 + \lambda^3(p-1) - \lambda^2\left(p(p-1)^2\right) - \lambda\left(p(p-1)^3 + (p+1)(p^2-p+1)\right) + p^2(p-1)^3(2p-1) = 0$.*

Proof. Let $\mathbb{Z}_p \times \mathbb{Z}_{p^3}$ be a ring. Note that $\mathbb{Z}_p$ has no non-zero zero divisor and in $\mathbb{Z}_{p^3}$ the non-zero zero divisors are multiples of p and p^2.
$Z^*(\mathbb{Z}_p \times \mathbb{Z}_{p^3}) = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$, where $A_1 = \{(0, x_2) \,|x_2 \in \mathbb{Z}_{p^3}^*$ *and x_2 is non zero divisor* $\}$, $A_2 = \left\{(x_1, k_2 p^2) \,|x_1 \in \mathbb{Z}_p^* \,\&\, k_1 = 1,2,3,...,p-1\right\}$, $A_3 =$

$\{(x_1,k_1p)\,|x_1\in\mathbb{Z}_p^*\ \&\ k_2=1,2,3,...,p^2-1\ and\ p\nmid k_1\}$, $A_4=\{(x_1,0)\,|x_1\in\mathbb{Z}_p^*\}$, $A_5=\{(0,k_1p)\,|k_2=1,2,3,...,p^2-1\ and\ p\nmid k_1\}$ and $A_6=\{\left(0,k_2p^2\right)|k_1=1,2,3,...,p-1\}$.

Then $|A_1|=p^2(p-1)$, $|A_2|=(p-1)^2$, $|A_3|=p\,(p-1)^2$, $|A_4|=(p-1)$, $|A_5|=p(p-1)$, & $|A_6|=(p-1)$. Since all the elements of A_1 is adjacent with all the elements of A_4, we get the matrix of ones. Similarly we get the matrices of ones corresponding to A_2 & A_5, A_2 & A_6, A_3 & A_6, A_4 & A_5;A_4 & A_6; and A_5 & A_6. As no element of A_1 is adjacent with element of A_1, A_2, A_3,A_5, A_6, we get zero matrix. Similarly we get the zero matrices corresponding to rest of the pairs.

Hence we get adjacency matrix of zero divisor graph of $\mathbb{Z}_p\times\mathbb{Z}_{pq}$ by considering A_1 first, then A_2, then A_3, then A_4, then A_5 and then A_6 as $M(\Gamma(\mathbb{Z}_p\times\mathbb{Z}_{pq}))=$
$$\left[\begin{array}{c|c}[O]_{(p-1)(2p-1)\times(p-1)(2p-1)} & [S_1]_{(p-1)(2p-1)\times(p-1)(p+2)}\\ \hline [S_1^T]_{(p-1)(p+2)\times(p-1)(2p-1)} & [S_2]_{(p-1)(p+2)\times(p-1)(p+2)}\end{array}\right],$$
where O is the zero matrix of order $(p-1)(2p-1)\times(p-1)(2p-1)$ and

$$S_1=\begin{bmatrix} J & O & O\\ O & J & J\\ O & O & J\end{bmatrix}, S_2=\begin{bmatrix} O & J & J\\ J & O & J\\ J & J & J\end{bmatrix}$$

Let λ be any non-zero eigenvalue of $M(\Gamma(\mathbb{Z}_p\times\mathbb{Z}_{p^3}))$, then $|\lambda I-M(\Gamma(\mathbb{Z}_p\times\mathbb{Z}_{p^3}))|=0$

Now let $T_1=\lambda I-O$, $T_2=S_1$, $T_3=S_1^T$ and $T_4=\lambda I-S_2$. Then by Proposition 2.1 $|\lambda I-M(\Gamma(\mathbb{Z}_p\times\mathbb{Z}_{p^3}))|=\begin{vmatrix}T_1 & T_2\\ T_3 & T_4\end{vmatrix}=|T_1||T_4-T_3T_1^{-1}T_2|=0$.

Since T_1 is a scalar matrix of order $(p-1)(2p-1)$, we get $|T_1|=\lambda^{(p-1)(2p-1)}$ and
$$T_3T_1^{-1}T_2=\begin{bmatrix}\frac{p^2(p-1)}{\lambda}J & O & O\\ O & \frac{(p-1)^2}{\lambda}J & O\\ O & O & \frac{(p-1)^2(p+1)}{\lambda}J\end{bmatrix}$$
and $|T_4-T_3T_1^{-1}T_2|=\lambda^{(p-1)(p+2)-4}(\lambda^4+\lambda^3(p-1)-\lambda^2\left(p\,(p-1)^2\right)-\lambda\left(p\,(p-1)^3+(p+1)(p^2-p+1)\right)+p^2\,(p-1)^3\,(2p-1))$.

Therefore $|\lambda I-M(\Gamma(\mathbb{Z}_p\times\mathbb{Z}_{p^3}))|=\lambda^{(p-1)(2p-1)+(p-1)(p+2)-4}(\lambda^4+\lambda^3(p-1)-\lambda^2\left(p\,(p-1)^2\right)-\lambda\,(p\,(p-1)^3+(p+1)(p^2-p+1))+p^2\,(p-1)^3\,(2p-1))=0$ which is the characteristic polynomial.

Since $\lambda\neq 0$, we get $\lambda^4+\lambda^3(p-1)-\lambda^2\left(p\,(p-1)^2\right)-\lambda\left(p\,(p-1)^3+(p+1)(p^2-p+1)\right)+p^2\,(p-1)^3\,(2p-1))=0$ □

Conclusion

We have explored the concept of graph energy in the context of zero divisor graphs and obtained characteristic equations for various graphs. We have also investigated the energy of the graph $\mathbb{Z}_p\times\mathbb{Z}_p$ (where p is prime.)

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