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# ON EDGE-EDGE DOMINATION IN GRAPHS

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***ABSTRACT***

*In this paper we further study the concept of edge-edge domination in graphs. We observe that the edge-edge domination number of a graph may increase or decrease or remains same when a edge is removed from a graph. We proved a necessary and sufficient condition under which the edge-edge domination number of a graph increases and also we proved a necessary and sufficient condition under which the edge-edge domination number of a graph decreases. For this purpose we introduce two new concepts namely e-dominating neighbourhood of an edge and private edge-edge neighbourhood of an edge with respect to a set containing the edge. Some examples also have been given.*

***Key words:*** *edge-edge dominating set, minimal edge-edge dominating set, minimum edge-edge dominating set, edge-edge domination number, e-dominating neighbourhood, Private edge-edge neighbourhood.*

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## INTRODUCTION

The concept of edge-edge domination was introduced by R. S. Bhat, S. S. Kamath and S. R. Bhat in [4]. An edge $g = uv$ be e-dominates the edge $h = xy$ if $x, y \in N[u] \cup N[v]$. A set $F$ of edges is said to be an edge-edge dominating(EED) set of $G$ if for every edge $h$ not in $F$, there is an edge $g$ in $F$ such that $h$ is e-dominated by $g$. An edge-edge dominating set with minimum cardinality is called minimum edge-edge dominating set. The cardinality of minimum edge-edge dominating set is called edge-edge domination number and it is denoted by $\gamma_{ee}(G)$ [4].

It may happens that $g$ e-dominates $h$ but $h$ does not e-dominate $g$. We may note that if $F$ is an edge dominating set then every edge $g$ which is not in $F$ is adjacent to some member of $F$ and thus $F$ is an EED set of $G$. However, an EED set need not be edge dominating set.

The concept of vertices dominates edges and edges dominate vertices was introduced in 1985 by R. Laskar and Ken Peters [3] and then in 1992 by, Sampathkumar and S. S. Kamath [2]. A vertex $v$ of a graph $G$ m-dominates an edge $xy$ if $xy$ is an edge of the subgraph induced by the vertices of the $N[v]$. An edge $x$ m-dominates a vertex $v$ if $v \in N[x]$. Suppose $xy$ & $uv$ are two edges and suppose $x$ m-dominates the edge $uv$ then obviously $xy$ e-dominates $uv$. Suppose $xy$ and $uv$ are two edges and suppose edge $xy$ m-dominates $u$ and $xy$ m-dominates $v$ then obviously $xy$ e-dominates $uv$ (edge). Converse is also true. Also note that $\gamma_{ee}(G) \le \gamma'(G)$.

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**PRELIMINARIES AND NOTATIONS**

If $G$ is a graph then $E(G)$ denotes the edge set and $V(G)$ denotes the vertex set of the graph. $S$ is any set then $|S|$ denotes the cardinality of $S$ and $E(G)\setminus S$ is a subgraph of $G$ obtained by removing the edges of $S$. If $f$ is an edge of $G$ then $G\setminus f$ denotes the subgraph of $G$ obtained by removing the edge $f$. $N[x]$ denotes the set of adjacent vertices of $v$ including $v$ and $N(V)$ denotes the set of vertices which are adjacent to $v$.

In this paper we consider only simple graphs with finite vertex set.

**Proposition 1:** Let $G$ be a graph and $g = uv$ be an isolated edge of $G$. Let $F$ be any EED set of $G$ then $g \in F$.

**Proof:** Suppose $g \notin F$. Now there is an edge $f = xy \in F$ such that $g$ is e-dominated by $f$ and therefore $u = x$ or $u$ is adjacent to $x$ and $v = y$ or $v$ is adjacent to $y$. But this implies that $g$ is not an isolated edge of $G$ and therefore $g \in F$.

**Proposition 2:** Let $G$ be a graph. $g$ be a pendant edge of $G$ and $F$ be a minimum EED set of $G$ then $g \in F$ or for some edge $f$ adjacent to $g$, $f \in F$.

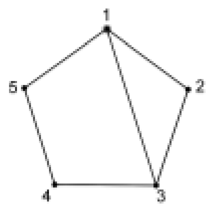
**Proof:** Suppose $g \notin F$. Now $e$ is e-dominated by some edge $f$ in $F$. Let $g = uv$ and $f = xy$. Suppose, $v$ is the pendant vertex of $g$.

**Case-I :** $u = x$ or $u = y$
Then obviously $f = xy$ is adjacent to $g$.

**Case II:** $u \neq x$, $u \neq y$
Then $v \notin N[x] \cup N[y]$ and this contradicts the fact that $g$ is e-dominated by $f = xy$. Thus $g$ is adjacent to $f = xy$.

**Definition 3:** (e-dominating neighbourhood) Let $G$ be a graph and $f$ be an edge of $G$. Then e-dominating neighbourhood of $f$ in $G$ is the set $N_{ee}[f] = \{g \in E(G) / g \text{ e-dominates } f\}$.

**Example 1:** Consider the graph $G_1$. Let $f = \{15\}$ then the e-dominating neighbourhood of $f$ will be the set $\{12,\ 13,\ 45\}$.




$G_1$

**Theorem 4:** Let $G$ be a graph and $f$ be any edge of $G$. Then $\gamma_{ee}(G\setminus f) > \gamma_{ee}(G)$ iff

1. $f$ is not an isolated edge of $G$.
2. For every minimum EED set $F$ of $G$, $f \in F$.
3. There is no subset $F$ of $E(G)\setminus N_{ee}[f]$ such that $|F| \leq \gamma_{ee}(G)$ and $F$ is an EED set of $G\setminus f$.

**Proof:** First suppose that $\gamma_{ee}(G\setminus f) > \gamma_{ee}(G)$.

1. Suppose $f$ is an isolated edge of $G$. Let $F$ be any minimum EED set of $G$. Then $f \in F$. Let $F_1 = F\setminus\{f\}$. Now consider the subgraph $G\setminus f$. Let $h$ be any edge of $G\setminus f$ such that $h$ does not belongs to $F_1$. Then $h \notin F$. Now $h$ is e-dominated by some member $g$ of $F$. Obviously, $g \neq f$ because $f$ is isolate edge of $G$. Therefore $h$ is e-dominated by some member of $F_1$. Thus $F_1$ is EED set of $G\setminus f$.Hence, $\gamma_{ee}(G\setminus f) \leq |F_1| < |F| = \gamma_{ee}(G)$. Which is a contradiction. Therefore $f$ is not isolated edge of $G$.

2. Suppose there is a minimum EED set $F$ of $G$ such that $f \notin F$. Now, consider the subgraph $G \setminus f$. It can be easily proved that $F$ is an EED set of $G \setminus f$. Therefore, $\gamma_{ee}(G \setminus f) < |F| = \gamma_{ee}(G)$. Which is a contradiction. Therefore, condition (2) holds.
3. Suppose there is a subset $F$ of $E(G) \setminus N_{ee}[f]$ and $F$ is an EED set of $G \setminus f$. Then again $\gamma_{ee}(G \setminus f) \leq |F| \leq \gamma_{ee}(G)$. Which is again a contradiction. Therefore, condition (3) also holds.

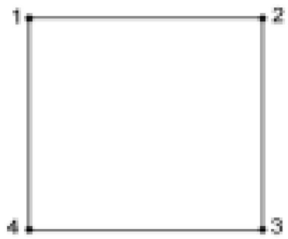
Conversely, suppose (1), (2) & (3) hold.

First suppose that $\gamma_{ee}(G \setminus f) = \gamma_{ee}(G)$. Let $F$ be any minimum EED set of $G \setminus f$. Suppose $F$ is also an EED set of $G$. Then $F$ is a minimum EED set of $G$ not containing $f$. Which contradicts (2).

Suppose $F$ is not an EED set of $G$. Therefore, $f$ is not e-dominated by any member of $F$ and therefore, $F \cap N_{ee}[f] = \phi$. Thus $F$ is an EED set of $G \setminus f$ such that $|F| \leq \gamma_{ee}(G)$ and $F$ is subset of $E(G) \setminus N_{ee}[f]$. This contradicts condition (3). Thus, in both the cases we have contradiction. Therefore, $\gamma_{ee}(G \setminus f) = \gamma_{ee}(G)$ is not possible.
Suppose, $\gamma_{ee}(G \setminus f) < \gamma_{ee}(G)$. Let $F$ be any minimum EED set of $G \setminus f$ then $F$ cannot be an EED set of $G$. Therefore, $f$ is not e-dominated by any member of $F$. Which means that $F$ is a subset of $E(G) \setminus N_{ee}[f]$. Also, $|F| \leq \gamma_{ee}(G)$ and $F$ is an EED set of $G \setminus f$. This again contradicts (3). Thus $\gamma_{ee}(G \setminus f) < \gamma_{ee}(G)$ is also not possible. Hence, it must be true that $\gamma_{ee}(G \setminus f) > \gamma_{ee}(G)$.

**Definition 5:** (Private edge-edge neighbourhood of $g$ with respect to $F$) Let $G$ be a graph. $F$ be a set of edges and $g \in F$. Then the private edge-edge neighbourhood of $g$ with respect to $F$ is $prn_{ee}[g,F]$ = { $h \in E(G) / h$ is e-dominated by only one member of $F$ namely $g$ } = $\{h \in E(G) / N_{ee}[h] \cap F = \{g\}\}$

**Example 2:** Consider the graph $G_2$. If we take $F = \{12, 34\}$ and $g = \{12\}$ then the private edge-edge neighbourhood of $g$ with respect to $F$ is the set $\{12\}$.




$G_2$

**Theorem 6:** Let $G$ be a graph and $g \in E(G)$. Then $\gamma_{ee}(G \setminus g) < \gamma_{ee}(G)$ if and only if there is a minimum EED set $F$ of $G$ containing $g$ such that $prn_{ee}[g,F] = \{g\}$.

**Proof:** Suppose, $\gamma_{ee}(G \setminus g) < \gamma_{ee}(G)$. Let $F_1$ be a minimum EED set of $G \setminus g$. Then $F_1$ cannot be an EED set of $G$. This implies that there is no member of $F_1$ which e-dominates $g$.

Let $F = F_1 \cup \{g\}$. Then obviously $F$ is an EED set of $G$. Since $\gamma_{ee}(G \setminus g) < \gamma_{ee}(G)$, $F$ must be a minimum EED set of $G$. Also, $g \in F$. Since $g$ is not e-dominated by any other member of $F$ and $g$ is e-dominated by $g$ itself, $g \in prn_{ee}[g,F]$ .Suppose $h$ is an edge of $G$ such that $h \neq g$ and $h \in prn_{ee}[g,F]$. Then, $h \notin F_1$ because if $h \in F_1$ then $h \in F$ and this implies that $h$ is e-dominated by two distinct member of $F$ namely $g$ and $h$ which is a contradiction and thus $h \notin F_1$.

Now, $h$ is e-dominated by some member $h'$ of $F_1$ because $F_1$ is a EED set of $G \setminus g$. Then $h' \in F$ and we have $h$ is e-dominated by two distinct members of $F$ namely $h'$ and $g$. Which is a contradiction. Thus, we have proved that if $h \neq g$ then $h \notin prn_{ee}[g,F]$. Thus, $prn_{ee}[g,F] = \{g\}$.

Conversely, suppose there is a minimum EED set $F$ of $G$ such that $prn_{ee}[g,F]=\{g\}$. Let $F_1 = F \setminus \{g\}$. Let $h$ be an edge of $G \setminus g$ such that $h \notin F_1$. Then $h \notin F$.

Suppose $h$ is e-dominated by $g$. Since $h \notin prn_{ee}[g,F]$, $h$ must be e-dominated by some $g' \in F \ni g' \neq g$. Then $g' \in F_1$ and $h$ is e-dominated by $g'$. If $h$ is not e-dominated by $g$ then $h$ must be e-dominated by some other member $h''$ of $G$. Obviously, $h'' \in F_1$. Thus $F_1$ is an EED set of $G \setminus g$. Therefore $\gamma_{ee}(G \setminus g) < \gamma_{ee}(G)$.

**Corollary 7:** Let $G$ be a graph and $g \in E(G)$. If $\gamma_{ee}(G \setminus g) < \gamma_{ee}(G)$ then $\gamma_{ee}(G \setminus g) = \gamma_{ee}(G) - 1$.

**Corollary 8:** Let $G$ be a graph, $g$ and $h$ be two edges of $G$ such that $\gamma_{ee}(G \setminus g) < \gamma_{ee}(G)$ and $\gamma_{ee}(G \setminus h) > \gamma_{ee}(G)$. Then $g$ is not e-dominated by $h$. In particular, $g$ and $h$ cannot be an adjacent edges.

**Proof:** There is a minimum EED set of $G$ such that $g \in F$ and $prn_{ee}[g,F]=\{g\}$. This means that $g$ is not e-dominated by any other member of $F$. In Particular, $g$ is not adjacent to any other member of $F$. Now, $h \in F$ by theorem ( $\gamma_{ee}(G \setminus h) > \gamma_{ee}(G)$ ). Therefore, $g$ is not e-dominated by $h$. Also $g$ and $h$ are non-adjacent edges.

**Theorem 9:** Let $G$ be a graph. $g$ be an edge of $G$ such that $\gamma_{ee}(G \setminus g) > \gamma_{ee}(G)$. If $F$ is a minimum EED set of $G$ then $g \in F$ and $prn_{ee}[g,F]$ contains two non-adjacent edges.

**Proof:** Since, $\gamma_{ee}(G \setminus g) > \gamma_{ee}(G)$ and $g \in F$. Since $F$ is a minimal EED of $G$, $prn_{ee}[g,F] \neq \phi$. If $prn_{ee}[g,F]=\{g\}$ then $\gamma_{ee}(G \setminus g) < \gamma_{ee}(G)$ (By theorem 6) and this is a contradiction. Therefore, there is an edge $h \neq g$ such that $h \in prn_{ee}[g,F]$. Obviously $h \notin F$. Suppose $prn_{ee}[g,F]=\{h\}$. Now consider the set $F_1 = (F \setminus \{g\}) \cup \{h\}$. Then $|F_1| = |F|$ and $g \notin F_1$.

Let $h'$ be any edge of $G$ such that $h' \in F_1$. Suppose $h' = g$. Now $g \notin prn_{ee}[g,F]$. Therefore, $h' = g$ is e-dominated by some member of $F_1$. Now suppose $h' \neq g$. Again $h' \notin prn_{ee}[g,F]$. Therefore $h'$ is e-dominated by some member of $F$ different from $g$. Therefore, $h'$ is e-dominated by some member of $F_1$. Thus $F_1$ is a minimum EED set of $G$ not containing $g$ which is a contradiction. Thus, $prn_{ee}[g,F]$ must contain at least two edges.

Suppose, $prn_{ee}[g,F]=\{g,h\}$, where $h \neq g$. If $g$ and $h$ are non-adjacent edges then the statement of the theorem is proved.

Suppose $g$ and $h$ are adjacent edges. Let $F_1 = (F \setminus \{g\}) \cup \{h\}$. Note that $|F_1| = |F|$. Since $g$ and $h$ are adjacent edges, $g$ is e-dominated by $h$ which is in $F_1$.

Let $f$ be any edge such that $f \notin F_1$ and $f \neq g$. If $f$ is e-dominated by $g$ then $f$ is also e-dominated by some other member of $F$ because $f \notin prn_{ee}[g,F]$. If $f$ is not e-dominated by $g$ then $f$ is also e-dominated by some other member of $F$. Thus $f$ is e-dominated by some member of $F_1$. Thus, $F_1$ is a minimum EED set of $G$ such that $g \notin F_1$ which is again a contradiction. Thus it follows that $prn_{ee}[g,F]$ contains at least two edges $h_1$ and $h_2$ such that $h_1 \neq g$, $h_2 \neq g$.

Suppose any two edges in the $prn_{ee}[g,F]$ are adjacent. Let $h_1, h_2 \in prn_{ee}[g,F]$ such that $h_1 \neq g$ and $h_2 \neq g$. Then $h_1$ and $h_2$ are adjacent.

Suppose $g \in prn_{ee}[g,F]$. If $h_1$ or $h_2$ is not adjacent to $g$ then we have a contradiction because $g$ & $h_1$ or $g$ & $h_2$ are non-adjacent edges in the $prn_{ee}[g,F]$. Therefore $g$ & $h_1$ are adjacent and $g$ & $h_2$ are adjacent. Let $F_1 = (F \setminus \{g\}) \cup \{h_1\}$. Obviously, $g$ is e-dominated by $h_1$ because $g$ & $h_1$ are adjacent. $h_2$ is e-dominated by $h_1$. Any other edge $f$ which is not in $F_1$, if it is e-dominated by $g$ or otherwise is e-dominated by some other member of $F$. This means that $f$ is e-dominated by some member of $F_1$. Thus we have proved that $F_1$ is a minimum EED set of $G$ not containing $g$ which is a contradiction. Thus we have proved that $g \notin prn_{ee}[g,F]$.

Again let $F_1 = (F \setminus \{g\}) \cup \{h_1\}$. Then $h_2$ is e-dominated by $h_1$. Since $g \notin prn_{ee}[g,F]$, $g$ is e-dominated by some member $f$ of $F$ which is also a member of $F_1$. Any other edge which is not in $F_1$ and if it is in the $prn_{ee}[g,F]$ then it is adjacent to $h_1$ and therefore it is e-dominated by some member of $F_1$. Any edge which is not in the $prn_{ee}[g,F]$ must be e-dominated by some member of $F_1$. Thus $F_1$ is minimum EED set of $G$ not containing $g$ which is a contradiction. Thus, we conclude that there are two distinct edges in the $prn_{ee}[g,F]$ which are non-adjacent edges.

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