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About ve-Domination in Graphs

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Abstract. The paper is about the ve-domination (vertex-edge domination) in graphs. Necessary and sufficient conditions are proved under which the ve-domination number decreases or increases.

Keywords: ve-dominating set, minimal ve-dominating set, minimum ve-dominating set, ve-domination number, edge private neighbourhood.

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1. Introduction

The domination related results have appeared in several articles like [1]. Generalizations of graphs like hypergraphs, semigraphs and others have also been considered by several authors [5,6]. Mixed domination provides a possibility of exploring the above structures further. The concept of vertices dominates edges and edges dominate vertices are studied by several authors. The concept of ve-domination was studied by Sampathkumar and others [2,4]. A vertex v of a graph G m-dominates an edge xy if xy is an edge of the subgraph induced by the vertices of the $N[v]$. A set S of vertices is said to be a ve-dominating set if every edge of the graph G is m-dominated by some vertex in S. This concept is well studied in [3].

In this paper, we study this concept in the context of an operation called the vertex removal from a graph. We characterize a minimal ve-dominating set of a graph and also prove necessary and sufficient conditions under which the ve-domination number of a graph increases or decreases.

2. Preliminaries and notations

If G is a graph then its vertex set will be denoted as $V(G)$. For any subset S of a set of vertices $V(G)$, $V(G)\setminus S$ is a subgraph of G obtained by removing the vertices of S and all the edges incident to the vertices of S. If v is a vertex of G then $G\setminus v$ denotes the subgraph of G obtained by removing the vertex v and all the edges incident to v. If $v \in V(G)$ then $N(v)$ = The set of all vertices adjacent to v and $N[v] = N(v) \cup \{v\}$.

We consider only those graphs which are simple, undirected and having finite vertex set.

Definition 1. (ve-dominating set) A set $S \subset V(G)$ is a *ve-dominating* set if every edge of G is m-dominated by a vertex in S.

Definition 2. (Minimal ve-dominating set) A ve-dominating set S for a graph G is said to be *minimal ve-dominating set* for G if no proper subset S' of S is a ve-dominating set for the graph G.

Definition 3. (Minimum ve-dominating set) A ve-dominating set of minimum cardinality is called *minimum ve-dominating set.*

Definition 4. (ve-domination number) The *ve-domination number* for the graph G is denoted by $\gamma_{ve}(G)$ and is the cardinality of a minimum ve-dominating set.

Definition 5. (Edge private neighbourhood of a vertex) Let G be a graph, $S \subset V(G)$ and $v \in S$. Then *edge private neighbourhood of* v *with respect to* S is $prne[v,S]$ ={ $e \in E(G)$ such that e is an edge of the induced subgraph of the $N[v]$ but e is not an edge of the induced subgraph of the closed neighbourhood of any other vertex of S }.

3. Main results

Theorem 6. Let G be a graph and $v \in V(G)$. Then $\gamma_{ve}(G \setminus v) < \gamma_{ve}(G)$ if and only if there is a minimum ve-dominating set S containing v such that $prne[v,S]$ is a non-empty subset of all T = The set of all edges incident at v.

Proof: Suppose $\gamma_{ve}(G \setminus v) < \gamma_{ve}(G)$. Therefore v is not an isolated vertex. Let S_1 be a minimum ve-dominating set of $G \setminus v$. Then S_1 cannot be a ve-dominating set of G. So, there is an edge f of G which is not m-dominated by any vertex of S_1. We may note that one end vertex of this edge must be v. Note that the other end vertex of this edge is not in S_1. Let $S = S_1 \cup \{v\}$. First we prove that S is a ve-dominating set. Let e be any edge of G. If e is an edge of $G \setminus v$ then e is m-dominated by some vertex of S_1. If v is an end vertex of e then e is m-dominated by v. Thus from both the above it follows that cases e is m-dominated by some vertex of S. Therefore S is a ve-dominating set. Since $|S| > |S_1|$, S is a minimum ve-dominating set of G and $v \in S$. Let $f \in prne[v,S]$. Suppose no end vertex of f is v. Therefore f is an edge of $G \setminus v$. Therefore f is m-dominated by some vertex u of S_1. This is a contradiction as $f \in prne[v,S]$. Therefore f is incident at v.

Conversely, suppose that there is a minimum set S containing v such that $prne[v,S]$ is a non-empty subset of T. Let $S_1 = S \setminus \{v\}$. Let f be any edge of $G \setminus v$.

Since no end vertex of f is equal to v, $f \notin T$.Therefore, $f \notin prne[v,S]$. So, either f is not m-dominated by v or if it is m-dominated by v then it is also m-dominated by some other vertex of S. Suppose f is not m-dominated by v. Since S is a ve-dominating set of G, f is m-dominated by some other vertex u of S. Then $u \in S_1$ and therefore f is m-dominated by some vertex of S_1. Suppose f is m-dominated by v. Then f must be m-dominated by some other vertex w of S. Since $w \neq v$, $w \in S_1$.Thus f is m-dominated by some vertex of S_1. Therefore, S_1 is a ve-dominating set of $G \setminus v$. Therefore, $\gamma_{ve}(G \setminus v) \leq |S_1| < |S| = \gamma_{ve}(G)$. Therefore, $\gamma_{ve}(G \setminus v) < \gamma_{ve}(G)$.

Corollary 7. Let G be a graph and $v \in V(G)$. If $\gamma_{ve}(G \setminus v) < \gamma_{ve}(G)$, then $\gamma_{ve}(G \setminus v) = \gamma_{ve}(G) - 1$.

Proof: Let S_1 be a minimum ve-dominating set of $G \setminus v$. Then S_1 cannot be a ve-dominating set of G. Let $S = S_1 \cup \{v\}$. Then S is a minimum ve-dominating set of G and $|S| = |S_1| + 1$. That is $\gamma_{ve}(G) = \gamma_{ve}(G \setminus v) + 1$. Therefore, $\gamma_{ve}(G \setminus v) = \gamma_{ve}(G) - 1$.

Remark 8. The above corollary is also true for any graph which does not contain a triangle. For example, for any cycle C_n with $n \geq 4$ this corollary is true.

In [3], Sampathkumar and others have mentioned that for a triangle free graph the concepts of vertex covering and ve-domination are the same.

Proposition 9. Let G be a graph which is a triangle free and let $v \in V(G)$. Let S be a minimum ve-dominating set of G such that $v \in S$ then $\gamma_{ve}(G \setminus v) < \gamma_{ve}(G)$.

Proof: Since S is a minimal ve-dominating set, $prne[v,S] \neq \phi$. Let $e \in prne[v,S]$. If $e = xy$ then it cannot be happen that $x \neq v$ and $y \neq v$ because this will gives rise to a triangle which cannot exist in G. Thus one end vertex of e must be v. Thus all the edges which are in the $prne[v,S]$ have one end vertex v. Therefore $\gamma_{ve}(G \setminus v) < \gamma_{ve}(G)$.

Corollary 10. Let T be a tree, S be a minimum ve-dominating set of T and let $v \in S$ then $\gamma_{ve}(T \setminus v) < \gamma_{ve}(T)$.

Proposition 11. Let T be a tree, v be a pendant vertex and u be its neighbour which is called a supporting vertex of v. Let S be a minimum ve-dominating set of T. Then exactly one of u and v belongs to S.

Proof: Suppose $u \notin S$, $v \notin S$. Then the edge uv is not m-dominated by any vertex of S because T is a tree and therefore it does not contain a triangle. Therefore,

$u \in S$ or $v \in S$. Suppose, $u \in S$ and $v \in S$. Since S is a minimal ve-dominating set, every vertex in S must have a private edge neighbour but v does not have any private edge neighbour as $u \in S$. Thus we have a contradiction. Therefore, either $u \in S$ and $v \notin S$ or $v \in S$ and $u \notin S$.

Corollary 12. Let T be a tree, v be a pendant vertex and u be its supporting vertex. Then, $\gamma_{ve}(T \setminus v) < \gamma_{ve}(T)$.

Proof: We need to show that there is a minimum ve-dominating set such that $u \in S$. Let S be a minimum ve-dominating set of T and suppose, $u \notin S$. Then, $v \in S$. Let $S_1 = (S \setminus \{v\}) \cup \{u\}$. Then S_1 is a minimum ve-dominating set of T containing u. Therefore, $\gamma_{ve}(T \setminus u) < \gamma_{ve}(T)$.

Remark 13. Consider the cycle C_n, if n is odd then its ve-domination number is $\frac{n+1}{2}$. If we remove any vertex from this cycle then we get a path with $n-1$ vertices and $n-1$ is even and its ve-domination number is $\frac{n-1}{2}$. Thus ve-domination number decreases. Similarly, if n is an even the its ve-domination number is $\frac{n}{2}$. If we remove any vertex from this cycle then we get a path with $n-1$ vertices which is an odd number and its ve-domination number is $\frac{n-2}{2}$. Thus, ve-domination number decreases in this case also. Thus we conclude that if C_n is cycle with $n \geq 4$ then for every vertex v, $\gamma_{ve}(C_n \setminus v) < \gamma_{ve}(C_n)$.

Theorem 14. Let G be graph and $v \in V(G)$. Then $\gamma_{ve}(G \setminus v) > \gamma_{ve}(G)$ if and only if following three conditions are satisfied.

(1) v is not an isolated vertex of G.

(2) $v \in S$, for every minimum ve-dominating set S of G.

(3) There is no subset S of $G \setminus v$ such that $N(v)$ intersects $V(G) \setminus S$ with $|S| \leq \gamma_{ve}(G)$ and S is a ve-dominating set of $G \setminus v$.

Proof: First suppose that $\gamma_{ve}(G \setminus v) > \gamma_{ve}(G)$.

(1) If v is an isolated vertex of G. Then, $\gamma_{ve}(G \setminus v) = \gamma_{ve}(G)$ which is a contradiction. Therefore v is not an isolated of G.

(2) Suppose there is a minimum ve-dominating set S such that $v \notin S$. Then S is a ve-dominating set of $G \setminus v$ and therefore, $\gamma_{ve}(G \setminus v) \leq |S| \leq \gamma_{ve}(G)$, which is a contradiction. Thus $v \in S$, for every minimum ve-dominating set S of G.

(3) Suppose there is a subset S of $V(G)$ such that $N(v) \subseteq V(G) \setminus S$, $|S| \le \gamma_{ve}(G)$ and S is a ve-dominating set of $G \setminus v$. Then again $\gamma_{ve}(G \setminus v) \le |S| \le \gamma_{ve}(G)$, which is a contradiction. Therefore condition (3) is satisfied.

Conversely, suppose condition (1), (2) and (3) are satisfied. First suppose that $\gamma_{ve}(G \setminus v) = \gamma_{ve}(G)$. Let S be any minimum ve-dominating set of $G \setminus v$. First suppose that S is a ve-dominating set of G. Then S is a minimum ve-dominating set of G and $v \notin S$, which contradicts condition (2). Thus S is not a ve-dominating set of G. Therefore there is a neighbour u of v such that $u \notin S$. Then $N(v) \cap (V(G) \setminus S) \ne \phi$ and S is a ve-dominating set of $G \setminus v$ with $|S| \le \gamma_{ve}(G)$. This contradicts condition (3). Thus $\gamma_{ve}(G \setminus v) = \gamma_{ve}(G)$ is not possible. Suppose, $\gamma_{ve}(G \setminus v) < \gamma_{ve}(G)$. Let S be a minimum ve-dominating set of $G \setminus v$. Since, $|S| < \gamma_{ve}(G)$. Therefore S cannot be a ve-dominating set of G. Therefore $N(v)$ is not a subset of S and thus $N(v) \cap (V(G) \setminus S) \ne \phi$, $|S| \le \gamma_{ve}(G)$ and S is a ve-dominating set of $G \setminus v$, which is a contradiction. Therefore $\gamma_{ve}(G \setminus v) < \gamma_{ve}(G)$ is also not possible. Therefore $\gamma_{ve}(G \setminus v) > \gamma_{ve}(G)$.

Theorem 15. Let G be a graph, $v \in V(G)$ and suppose, $\gamma_{ve}(G \setminus v) > \gamma_{ve}(G)$. If S is a minimum ve-dominating set of G then $v \in S$ and $prne[v,S]$ is contain at least two non-adjacent edges.

Proof: Since $\gamma_{ve}(G \setminus v) > \gamma_{ve}(G)$, by condition(2) of theorem 14, $v \in S$. Also S is a minimal ve-dominating set of G and therefore, $prne[v,S] \ne \phi$. If all the edges in the $prne[v,S]$ are incident at v then it follows that $\gamma_{ve}(G \setminus v) < \gamma_{ve}(G)$, which is a contradiction. Therefore, there is an edge $xy \ni x \ne v, y \ne v$ and $xy \in prne[v,S]$. Suppose xy is the only edge such that $xy \in prne[v,S]$ and $x \ne v, y \ne v$. Let $S_1 = (S \setminus \{v\}) \cup \{x\}$. Let e be any edge of G. If e is not m-dominated by v then e is m-dominated by some vertex z in S such that $z \ne v$. Then $z \in S_1$ and e is m-dominated by z. Suppose e is again any edge of G. Suppose e is m-dominated by v but $e \notin prne[v,S]$. Then e is m-dominated by some vertex $w \in S \ni w \ne v$. Then again it is clear that e is m-dominated by some vertex of S_1. Let e be any edge of G such that if $e \in prne[v,S]$ and $e \notin \{xy\}$. Therefore one end vertex of e must be v. Suppose that other end vertex of e is equal to x then e is m-dominated by some vertex (namely x) of S_1. If the other end vertex of e is equal to y then $e = vy$ and then e is m-dominated by x which is in S_1. Thus, we have proved that if e is any edge of G then e is m-dominated by some member of S_1. Therefore S_1 is a minimum ve-dominating set of G

such that $v \notin S$, which is a contradiction. Thus apart from xy there is another edge f such that none of its end vertex is v and $f \in prne[v,S]$.

Suppose any two edges which are in the $prne[v,S]$ are adjacent. Let x_1y_1 and x_2y_2 be two edges which are in the $prne[v,S]$ and which do not have v as an end vertex. Now, they are adjacent edges. Suppose $x_1 = x_2$ and $y_1 \neq y_2$. Let $S_1 = (S \setminus \{v\}) \cup \{x_1\}$. Let f be any edge of G. If f is not m-dominated by v or f is not in the $prne[v,S]$ then f is m-dominated by some vertex of S_1. Suppose f is in the $prne[v,S]$. First suppose v is an end vertex of f. Let w be the other end vertex of f. If $w \in \{x_1, x_2, y_1, y_2\}$ then f is m-dominated by x_1. Suppose $w \notin \{x_1, x_2, y_1, y_2\}$ then $f = vw$ is not adjacent to the edge x_1y_1 and both these edges are in $prne[v,S]$ which is a contradiction. Thus v cannot be an end vertex of f then $f = zw$, for some vertex $z \neq v$. Then f is an edge of G such that f is in the $prne[v,S]$ and none of its end vertex is v. Now $f = zw$ is adjacent to x_1y_1 and it is also adjacent to x_2y_2. Therefore, $z, w \in \{y_1, y_2\}$. Therefore, zw is m-dominated by x_1. Thus every edge of G is m-dominated by some vertex of S_1. Thus S_1 is a minimum ve-dominating set with $v \notin S_1$, which is a contradiction. Thus the theorem is proved.

4. Concluding remark

In this paper, there is no restriction on the induced subgraph of the ve-dominating set. We may get new variants of ve-domination by requiring that the ve-dominating set is either an independent set or without isolated vertices or having isolate vertex and so on. Different condition will provide new directions for ve-domination in graphs.

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