MATCH
Communications in Mathematical and in Computer Chemistry

MATCH Commun. Math. Comput. Chem. **77** (2017) 589-594

ISSN 0340 - 6253

Some New Results on Energy of Graphs

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(Received August 26, 2016)

Abstract

The eigenvalue of a graph G is the eigenvalue of its adjacency matrix. The energy $E(G)$ of G is the sum of absolute values of its eigenvalues. A natural question arises: How the energy of a given graph G can be related with the graph obtained from G by means of some graph operations? In order to answer this question, we have considered two graphs namely, splitting graph $S'(G)$ and shadow graph $D_2(G)$. It has been proven that $E(S'(G)) = \sqrt{5}\,E(G)$ and $E(D_2(G)) = 2E(G)$.

1 Introduction

All graphs considered here are simple, finite and undirected. For standard terminology and notations related to graph theory, we follow Balakrishnan and Ranganathan [2] while for algebra we follow Lang [7].

The adjacency matrix $A(G)$ of a graph G with vertices $v_1, v_2, \ldots, v_n$ is an $n \times n$ matrix $[a_{ij}]$ such that, $a_{ij} = 1$ if v_i is adjacent to v_j, and 0 otherwise.

The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the graph G are the eigenvalues of its adjacency matrix. The set of eigenvalues of the graph with their multiplicities is known as spectrum of the graph.

In 1978 Gutman [5] defined the energy of a graph G as the sum of absolute values of the eigenvalues of graph G and denoted it by $E(G)$. Hence,

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

In 2004, Bapat and Pati [3] proved that if the energy of a graph is rational then it must be an even integer, while Pirzada and Gutman [9] established that the energy of a graph is never the square root of an odd integer. A brief account of graph energy is given in [1] as well as in the books [4,8]. Some fundamental results on graph energy are also reported in the thesis of Sriraj [10].

The present work is intended to relate the graph energy to larger graphs obtained from the given graph by means of some graph operations.

Let $A \in R^{m\times n}$, $B \in R^{p\times q}$. Then the *Kronecker product* (or tensor product) of A and B is defined as the matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

Proposition 1.1. [6] *Let $A \in M^m$ and $B \in M^n$. Furthermore, let λ be an eigenvalues of matrix A with corresponding eigenvector x, and μ an eigenvalue of matrix B with corresponding eigenvector y. Then $\lambda\mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $x \otimes y$.*

2 Energy of Splitting Graph

The *splitting graph* $S'(G)$ of a graph G is obtained by adding to each vertex v a new vertex v', such that v' is adjacent to every vertex that is adjacent to v in G. We prove the following result.

Theorem 2.1. $E(S'(G)) = \sqrt{5}\, E(G)$.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of the graph G. Then its adjacency matrix is given by

$$A(G) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \vdots \\ \boldsymbol{v_n} \end{array} \begin{array}{c} \begin{array}{cccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \cdots & \boldsymbol{v_n} \end{array} \\ \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix} \end{array}$$

Let $u_1, u_2, \ldots, u_n$ be the vertices corresponding to $v_1, v_2, \ldots, v_n$, which are added in G to obtain $S'(G)$, such that, $N(v_i) = N(u_i)$, $i = 1, 2, \ldots, n$. Then $A(S'(G))$ can be written as a block matrix as follows

$$A(S'(G)) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \vdots \\ \boldsymbol{v_n} \\ \\ \boldsymbol{u_1} \\ \boldsymbol{u_2} \\ \boldsymbol{u_3} \\ \vdots \\ \boldsymbol{u_n} \end{array} \begin{array}{c} \begin{array}{cccccccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \cdots & \boldsymbol{v_n} & \boldsymbol{u_1} & \boldsymbol{u_2} & \boldsymbol{u_3} & \cdot & \boldsymbol{u_n} \end{array} \\ \left[\begin{array}{ccccc:ccccc} 0 & a_{12} & a_{13} & \cdots & a_{1n} & 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} & a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & 0 & \cdots & a_{3n} & a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 & a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \\ \hdashline 0 & a_{12} & a_{13} & \cdots & a_{1n} & 0 & 0 & 0 & \cdots & 0 \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} & 0 & 0 & 0 & \cdots & 0 \\ a_{31} & a_{32} & 0 & \cdots & a_{3n} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \end{array}\right] \end{array}$$

That is,

$$A(S'(G)) = \begin{bmatrix} A(G) & A(G) \\ A(G) & O \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \otimes A(G)\,.$$

Hence,

$$spec(S'(G)) = \begin{pmatrix} \left(\dfrac{1+\sqrt{5}}{2}\right)\lambda_i & \left(\dfrac{1-\sqrt{5}}{2}\right)\lambda_i \\ n & n \end{pmatrix}$$

where λ_i , $i = 1, 2, \ldots, n$, are the eigenvalues of G, while $\left(\dfrac{1\pm\sqrt{5}}{2}\right)$ are the eigenvalues of $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Here,

$$E(S'(G)) = \sum_{i=1}^{n} \left| \left(\frac{1\pm\sqrt{5}}{2}\right)\lambda_i \right| = \sum_{i=1}^{n} |\lambda_i| \left[\frac{\sqrt{5}+1}{2} + \frac{\sqrt{5}-1}{2}\right]$$
$$= \sqrt{5}\sum_{i=1}^{n} |\lambda_i| = \sqrt{5}\,E(G)\,.$$

Illustration 2.2. *Consider the cycle C_4 and its spitting graph $S'(C_4)$. It is obvious that $E(C_4) = 4$ as $spec(C_4) = \begin{pmatrix} -2 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.*

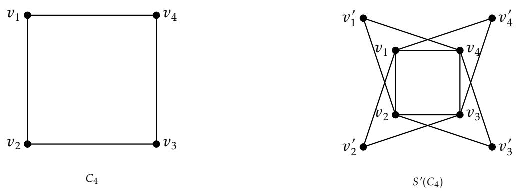

Figure 1

$$A(S'(C_4)) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \boldsymbol{v_4} \\ \boldsymbol{v'_1} \\ \boldsymbol{v'_2} \\ \boldsymbol{v'_3} \\ \boldsymbol{v'_4} \end{array} \begin{array}{c} \begin{matrix} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \boldsymbol{v_4} & \boldsymbol{v'_1} & \boldsymbol{v'_2} & \boldsymbol{v'_3} & \boldsymbol{v'_4} \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Therefore, spec$(S'(C_4))=\begin{pmatrix} 1+\sqrt{5} & 1-\sqrt{5} & -1-\sqrt{5} & -1+\sqrt{5} & 0 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix}$.

Hence,

$$E(S'(C_4)) = 4\sqrt{5} = \sqrt{5}\,E(C_4)\,.$$

3 Energy of Shadow Graph

The *shadow graph* $D_2(G)$ of a connected graph G is constructed by taking two copies of G, say G' and G''. Join each vertex u' in G' to the neighbors of the corresponding vertex u'' in G''.

Theorem 3.1. $E(D_2(G)) = 2E(G)$.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of the graph G. Then its adjacency matrix is same as in the proof of Theorem 2.1. Consider the second copy of graph G with vertices $u_1, u_2, u_3, \ldots, u_n$ to obtain $D_2(G)$, such that, $N(v_i) = N(u_i)\,,\, i = 1, 2, \ldots, n$. Then $A(D_2(G))$ can be written as a block matrix as follows:

$$A(D_2(G)) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \vdots \\ \boldsymbol{v_n} \\ \\ \boldsymbol{u_1} \\ \boldsymbol{u_2} \\ \boldsymbol{u_3} \\ \vdots \\ \boldsymbol{u_n} \end{array} \begin{array}{c} \begin{matrix} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \cdots & \boldsymbol{v_n} & \boldsymbol{u_1} & \boldsymbol{u_2} & \boldsymbol{u_3} & \cdot & \boldsymbol{u_n} \end{matrix} \\ \left[\begin{array}{ccccc|ccccc} 0 & a_{12} & a_{13} & \cdots & a_{1n} & 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} & a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & 0 & \cdots & a_{3n} & a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 & a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \\ \hline 0 & a_{12} & a_{13} & \cdots & a_{1n} & 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} & a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & 0 & \cdots & a_{3n} & a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 & a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{array}\right] \end{array}$$

That is,

$$A(D_2(G)) = \begin{bmatrix} A(G) & A(G) \\ A(G) & A(G) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes A(G)\,.$$

Hence,
$spec((D_2(G)) = \begin{pmatrix} 0 & 2\lambda_i \\ n & n \end{pmatrix}$, where λ_i , $i = 1, 2, \ldots, n$, are the eigenvalues of G, while $0, 2$ are eigenvalues of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Here,

$$E(D_2(G)) = \sum_{i=1}^{n} |2\lambda_i| = 2\sum_{i=1}^{n} |\lambda_i| = 2E(G)\,.$$

Illustration 3.2. *Consider the cycle C_4 and its shadow graph $D_2(C_4)$. From the previous example it is known that $E(C_4) = 4$ and spec(C_4) = $\begin{pmatrix} -2 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.*

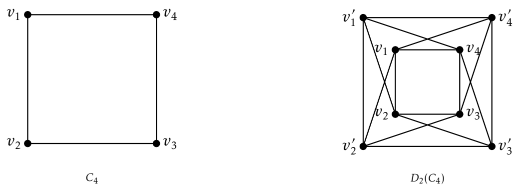

Figure 2

$$A(D_2(C_4)) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \boldsymbol{v_4} \\ \boldsymbol{v_1'} \\ \boldsymbol{v_2'} \\ \boldsymbol{v_3'} \\ \boldsymbol{v_4'} \end{array} \begin{array}{c} \begin{array}{cccccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \boldsymbol{v_4} & \boldsymbol{v_1'} & \boldsymbol{v_2'} & \boldsymbol{v_3'} & \boldsymbol{v_4'} \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Therefore, spec($D_2(C_4)$)=$\begin{pmatrix} 4 & -4 & 0 \\ 1 & 1 & 6 \end{pmatrix}$. Hence,

$$E(D_2(C_4)) = 8 = 2E(C_4)\,.$$

4 Concluding Remarks

The energy of a graph is one of the emerging concept within graph theory. This concept serves as a frontier between chemistry and mathematics. The energy of many graphs is known. But we have take up the problem to investigate the energy of a graph obtained by means of some graph operations on a given graph and it has been revealed that the energy of the new graph is a multiple of energy of a given graph.

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