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ENERGY OF *m*-SPLITTING AND *m*-SHADOW GRAPHS

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Abstract

We determine the energy of a graph obtained by means of graph operations on a given graph, and relate the energy of such a new graph with that of the given graph.

1. Introduction

For standard terminology and notations related to graph theory, we follow West [2] while for algebra we follow Lang [9].

Let G be a connected undirected simple graph with vertex set $V(G) = \{v_1, v_2, .., v_n\}$. The *adjacency matrix* of G, denoted by $A(G)$, is defined as $A(G) = [a_{ij}]$ such that $a_{ij} = 1$ if v_i is adjacent with v_j, and 0 otherwise.

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If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of G, then

$$spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ m_1 & m_2 & \cdots & m_n \end{pmatrix}.$$

The energy $E(G)$ of a graph G is defined to be the sum of all absolute values of eigenvalues of G. The concept of graph energy was introduced by Gutman [4] in 1978. A brief account of graph energy can be found in Cvetković et al. [3] and Li et al. [10].

This concept traces the connection in the study of approximation of the total π-electron energy of a conjugated hydrocarbon in molecular chemistry. A conjugated hydrocarbon can be represented by a graph called molecular graph in which every carbon atom is represented by a vertex, carbon-carbon bond by an edge and hydrogen atoms are ignored. The study of molecular structure with the help of energy of its graph is categorized as chemical graph theory.

The concepts like incidence energy [5], skew energy [1], distance energy [7] are also available in the literature.

Let $A \in R^{m \times n}$, $B \in R^{p \times q}$. Then the *Kronecker product* (or *tensor product*) of A and B is defined as the matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

Proposition 1.1 [6]. *Let $A \in M^m$ and $B \in M^n$. Furthermore, let λ be an eigenvalue of matrix A with corresponding eigenvector x and μ be an eigenvalue of matrix B with corresponding eigenvector y. Then $\lambda\mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $x \otimes y$.*

2. Energy of *m*-splitting Graph

Definition 2.1. The *splitting graph* $S'(G)$ of a graph G is obtained by adding to each vertex v a new vertex v' such that v' is adjacent to every vertex that is adjacent to v in G.

Definition 2.2. The *m-splitting graph* $Spl_m(G)$ of a graph G is obtained by adding to each vertex v of G new m vertices, say $v_1, v_2, v_3, \ldots, v_m$ such that v_i, $1 \leq i \leq m$ is adjacent to each vertex that is adjacent to v in G.

Theorem 2.3. $E(Spl_m(G)) = \sqrt{1+4m}\, E(G)$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the graph G. Then its adjacency matrix is given by

$$A(G) = \begin{array}{c} \\ \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{v}_3 \\ \vdots \\ \boldsymbol{v}_n \end{array} \begin{array}{c} \begin{array}{ccccc} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 & \cdots & \boldsymbol{v}_n \end{array} \\ \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix} \end{array}.$$

Let $v_i^1, v_i^2, \ldots, v_i^m$ be the vertices corresponding to each v_i, which are added in G to obtain $Spl_m(G)$ such that $N(v_i^1) = N(v_i^2) = \cdots = N(v_i^m) = N(v_i)$, $i = 1, 2, \ldots, n$. Then $A(Spl_m(G))$ can be written as a block matrix as follows:

$$A(Spl_m(G)) = \begin{bmatrix} A(G) & A(G) & \cdots & A(G) \\ A(G) & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ A(G) & O & \cdots & O \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}_{m+1} \otimes A(G).$$

Let $A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}_{m+1}$. Now we compute eigenvalues of matrix A.

Since matrix A is of rank two, A has two nonzero eigenvalues, say μ_1 and μ_2. Obviously,

$$\mu_1 + \mu_2 = tr(A) = 1. \tag{1}$$

Consider

$$A^2 = \begin{bmatrix} m+1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_{m+1}.$$

Here,

$$\mu_1^2 + \mu_2^2 = tr(A^2) = 2m + 1. \tag{2}$$

Solving two equations (1) and (2), we have

$$\mu_1 = \frac{1 + \sqrt{1 + 4m}}{2}, \quad \mu_2 = \frac{1 - \sqrt{1 + 4m}}{2}.$$

Hence,

$$spec(A) = \begin{pmatrix} 0 & \frac{1 + \sqrt{1 + 4m}}{2} & \frac{1 - \sqrt{1 + 4m}}{2} \\ m - 1 & 1 & 1 \end{pmatrix}.$$

Since $A(Spl_m(G)) = A \otimes A(G)$, it follows that if $\lambda_1, \lambda_2, ..., \lambda_n$ are eigenvalues of A, then by Proposition 1.1,

$spec(Spl_m(G)) =$

$$\begin{pmatrix} 0\lambda_1 & \cdots & 0\lambda_n & \left(\frac{1+\sqrt{1+4m}}{2}\right)\lambda_1 & \cdots & \left(\frac{1+\sqrt{1+4m}}{2}\right)\lambda_n & \left(\frac{1-\sqrt{1+4m}}{2}\right)\lambda_1 & \cdots & \left(\frac{1-\sqrt{1+4m}}{2}\right)\lambda_n \\ m-1 & \cdots & m-1 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Hence,

$$E(Spl_m(G)) = \sum_{i=1}^{n} \left| \left(\frac{1 \pm \sqrt{1+4m}}{2} \right) \lambda_i \right|$$

$$= \sum_{i=1}^{n} |\lambda_i| \left[\frac{\sqrt{1+4m}+1}{2} + \frac{\sqrt{1+4m}-1}{2} \right]$$

$$= \sqrt{1+4m} \sum_{i=1}^{n} |\lambda_i|$$

$$= \sqrt{1+4m}\, E(G).$$

The following illustration gives better understanding of Theorem 2.3.

Illustration 2.4. Consider cycle C_4 and $Spl_2(C_4)$. It is obvious that $E(C_4) = 4$ as $spec(C_4) = \begin{pmatrix} -2 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

Here, $spec(Spl_2(C_4)) = \begin{pmatrix} 2 & -2 & 4 & -4 & 0 \\ 1 & 1 & 1 & 1 & 8 \end{pmatrix}$.

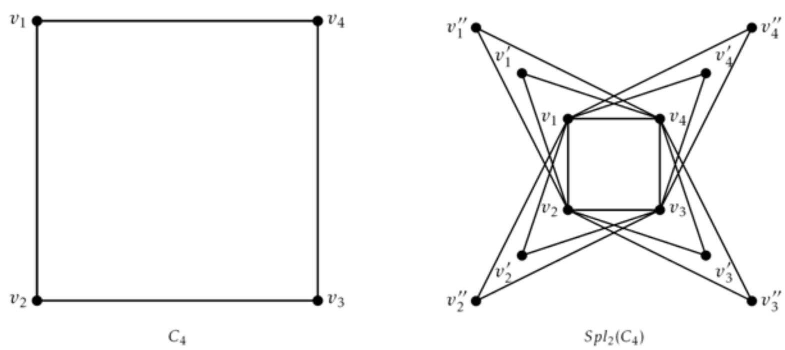

Figure 1

Hence,

$$E(Spl_2(C_4)) = 12 = \sqrt{1+4(2)}\, E(C_4).$$

Remark 2.5. For $m = 1$, the graph is called *splitting graph* denoted by $S'(G)$. It has been already proved by Vaidya and Popat [8] that $E(S'(G)) = \sqrt{5}E(G)$.

3. Energy of *m*-shadow Graph

Definition 3.1. The *shadow graph* $D_2(G)$ of a connected graph G is constructed by taking two copies of G, say G' and G''. Join each vertex u' in G' to the neighbors of the corresponding vertex u'' in G''.

Definition 3.2. The *m-shadow graph* $D_m(G)$ of a connected graph G is constructed by taking m copies of G, say $G_1, G_2, \ldots, G_m$, then join each vertex u in G_i to the neighbors of the corresponding vertex v in G_j, $1 \leq i, j \leq m$.

Theorem 3.3. $E(D_m(G)) = mE(G)$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the graph G. Then its adjacency matrix is same as in the proof of Theorem 2.3. Consider m copies of a graph G, say $G_1, G_2, \ldots, G_m$ with vertices $v_i^1, v_i^2, \ldots, v_i^m$, $1 \leq i \leq n$ to obtain $D_m(G)$ such that each vertex u in G_j is joined to the neighbors of the corresponding vertex v in G_k, $1 \leq j, k \leq m$.

Then the $A(D_m(G))$ can be written as a block matrix as follows:

$$A(D_m(G)) = \begin{bmatrix} A(G) & A(G) & \cdots & A(G) \\ A(G) & A(G) & \cdots & A(G) \\ \vdots & \vdots & \ddots & \vdots \\ A(G) & A(G) & \cdots & A(G) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_m \otimes A(G)$$

$$= J_m \otimes A(G).$$

Hence, by Proposition 1.1,

$$spec(D_m(G)) = \begin{pmatrix} 0\lambda_1 & \cdots & 0\lambda_n & m\lambda_1 & \cdots & m\lambda_n \\ m-1 & \cdots & m-1 & 1 & \cdots & 1 \end{pmatrix},$$

where λ_i are eigenvalues of G, while $0\ (m-1\,\text{times}),\ m$ are eigenvalues of J_m. Here,

$$E(D_m(G)) = \sum_{i=1}^{n} | m\lambda_i | = m\sum_{i=1}^{n} | \lambda_i | = mE(G).$$

The following illustration helps us to understand Theorem 3.3.

Illustration 3.4. Consider cycle C_4 and $D_3(C_4)$. From the previous example, it is known that $E(C_4) = 4$ and $spec(C_4) = \begin{pmatrix} -2 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

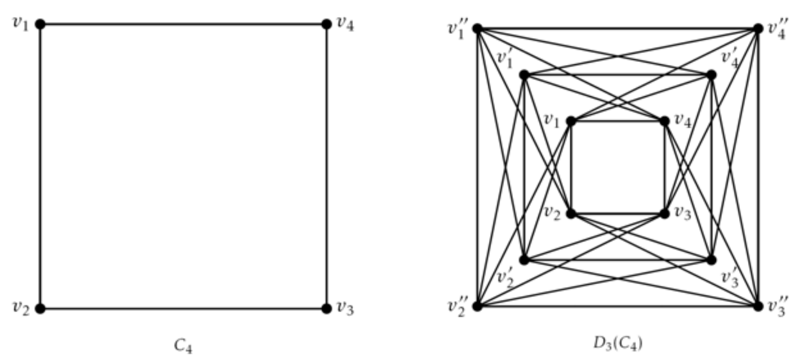

Figure 2

Here, $spec(D_3(C_4)) = \begin{pmatrix} 6 & -6 & 0 \\ 1 & 1 & 10 \end{pmatrix}$.

Hence,

$$E(D_3(C_4)) = 12 = 3(4) = 3E(C_4).$$

Remark 3.5. For $m = 1$, the graph is called *shadow graph* denoted by $D_2(G)$. It has been already proved by Vaidya and Popat [8] that $E(D_2(G)) = 2E(G)$.

4. Concluding Remarks

The energy of standard graphs is available in the literature but we have investigated the energy of larger graph obtained from a given graph by means of graph operations. We have obtained very general results by considering two graph operations called *m*-splitting and *m*-shadow graphs.

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