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Some New Results on Seidel Equienergetic Graphs

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Abstract. The energy of a graph G is the sum of the absolute values of the eigenvalues of the adjacency matrix of G. Some variants of energy can also be found in the literature, in which the energy is defined for the Laplacian matrix, Distance matrix, Common-neighbourhood matrix or Seidel matrix. The Seidel matrix of the graph G is the square matrix in which ij^{th} entry is -1 or 1, if the vertices v_i and v_j are adjacent or non-adjacent respectively, and is 0 , if $v_i = v_j$. The Seidel energy of G is the sum of the absolute values of the eigenvalues of its Seidel matrix. We present here some families of pairs of graphs whose Seidel matrices have different eigenvalues, but who have the same Seidel energies.

1. Introduction

For standard terminology and notations related to graph theory we follow Balakrishnan and Ranganathan [2] while for any undefined term in algebra we follow Lang [10]. Let G be a simple graph with vertex set $\{v_1, v_2, \cdots, v_n\}$. The *adjacency matrix* denoted by $A(G)$ of G is defined to be $A(G) = [a_{ij}]$, such that, $a_{ij} = 1$ if v_i is adjacent with v_j, and 0 otherwise.

The eigenvalues of A are called the eigenvalues of G. The energy $E(G)$ of graph G is the sum of all absolute values of eigenvalues of G. The concept of *energy of graph* was introduced by Gutman [5] in 1978. A brief account on energy of graph can be found in Cvetkovič [4] and Li [13].

The other varients of energy like Laplacian energy [7], Incidence energy [6], Skew

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energy [1], Distance energy [3], Seidel energy [8] are also available in the literature. In the present paper we have focused on Seidel energy of graphs.

Let G be a simple graph with vertex set $\{v_1, v_2, \cdots, v_n\}$. The *Seidel matrix* of G which is denoted by $S(G) = [s_{ij}]$ is a $n \times n$ matrix in which $s_{11} = s_{22} = \cdots = s_{nn} = 0$, and in which for $i \neq j$ we have $s_{ij} = -1$ if v_i is adjacent to v_j and $s_{ij} = 1$ otherwise.

The eigenvalues of the Seidel matrix, labeled as $\sigma_1, \sigma_2, \cdots, \sigma_n$, are said to be the Seidel eigenvalues of G. The collection of Seidel eigenvalues together with their multiplicities is known as Seidel spectrum of G denoted by $Spec_s(G)$. Haemers [8] has defined the Seidel energy of G as

$$SE(G) = \sum_{i=1}^{n} |\sigma_i|$$

As an example the, Seidel matrix of the complete graph K_n is $I - J$. Thus

$$Spec_s(K_n) = \{1^{n-1}, (1-n)^1\}$$

where a power denotes the multiplicity of an eigenvalue. Therefore, $SE(K_n) = 2n - 2$.

Two graphs G_1 and G_2 are said to be Seidel equienergetic if $SE(G_1) = SE(G_2)$. Of course, Seidel cospectral graphs are Seidel equienergetic. We are interested in graphs which are of same order, non co-spectral and equienergetic in the context of Seidel energy. Let $\overline{G}$ be the complement of the graph G then $S(G) = A(G) - A(\overline{G})$ implying $S(\overline{G}) = -S(G)$. Consequently, G and $\overline{G}$ are obviously Seidel equienergetic.

The *join of two graphs* G_1 and G_2, denoted by $G_1 + G_2$, is a graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to all vertices of G_2. Ramane *et al.* [11] proved that if H_1 and H_2 are Seidel non cospectral, Seidel equienergetic regular graphs on n vertices and of same degree, then for any regular graph G, $SE(H_1 + G) = SE(H_2 + G)$.

Ramane *et al.* [12] also proved that if G_1 and G_2 are two Seidel non-cospectral r-regular graphs of the same order and of the same degree $r \geq 3$, then for any $k \geq 2$, the iterated line graphs $L^k(G_1)$ and $L^k(G_2)$ are Seidel non-cospectral, Seidel equienergetic graphs.

2. Seidel Equienergetic Graphs

To state our results we need to defined some graphs. The *categorical product* $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ in which two vertices (u_1, v_1) and (u_2, v_2) are adjacent if $u_1 u_2 \in E(G)$ and $v_1 v_2 \in E(H)$.

Let $A \in R^{m \times n}$, $B \in R^{p \times q}$. The *Kronecker product* (or tensor product) of A and B is defined as the matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

It is well known and easy to show that for graphs G_1 and G_2 for which loops are allowed, we have

$$A(G_1 \times G_2) = A(G_1) \otimes A(G_2)$$

The following is useful for determining the spectrum of a product of graphs from those of its factors.

Proposition 2.1.([9]) *Let $A \in M^m$ and $B \in M^n$. Furthermore, let σ be eigenvalue of matrix A with corresponding eigenvector x and μ be eigenvalue of matrix B with corresponding eigenvector y. Then $\sigma\mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $x \otimes y$.*

Let G^r be the graph obtained from G by adding loops on every vertex and G^{ur} be the graph obtained from G by removing all loops. For any simple graph G, if we take $G_1 = K_m^r$ and $G_2 = G$, using the facts that $A(G^r) = A(G) + I, A(G^{ur}) = A(G) - I, A(K_m^r) = J$ and writing A for $A(G)$, we have,

$$A(K_m^r \times G) = J \otimes A$$

and

$$A(K_m^r \times G^r)^{ur} = J \otimes (A + I) - I \otimes I$$

We introduce two convenient notations. Let $D_m(G) = K_m^r \times G$ and $D_m^*(G) = (K_m^r \times G^r)^{ur}$. It is easy to verify that if G is a graph with n vertices then both $D_m(G)$ and $D_m^*(G)$ are graphs with mn vertices. The graphs $D_2(C_5)$ and $D_2^*(C_5)$ are shown in Figure 1.

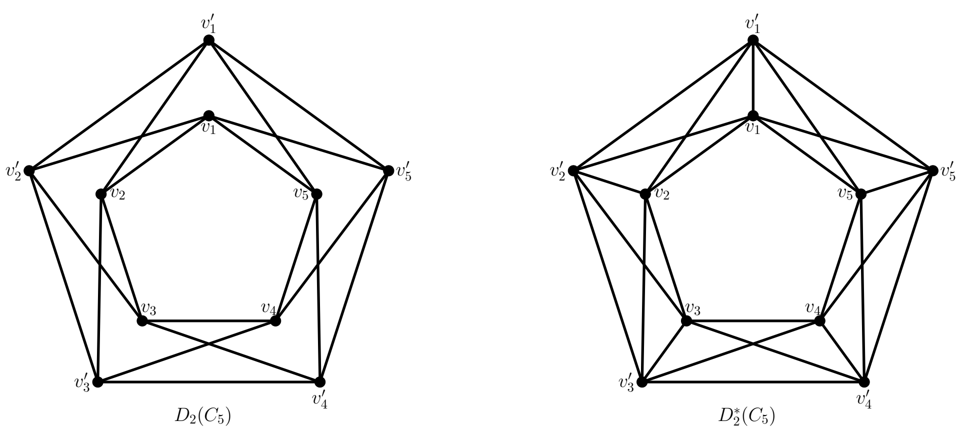

Figure 1: The graphs $D_2(C_5)$ and $D_2^*(C_5)$

Lemma 2.2. *If the Seidel spectrum of G is $\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ then the Seidel spectrum of $D_m(G)$ is*

$$Spec_s(D_m(G)) = \{m\sigma_i + (m-1) | \, i = 1, 2, \cdots, n\} \cup \{-1^{mn-n}\}$$

Proof. As $A(K_m^r) = J = J_m$ has spectrum $\{m, 0^{m-1}\}$, we have by Proposition 2.1 that an eigenvalue σ_i of $S(G)$ yields in $A(K_m^r) \times (S(G) + I) - I$ an eigenvalue $m(\sigma_i + 1) - 1 = m\sigma_i + (m-1)$ and $mn - n$ eigenvalue $0(\sigma_i + 1) - 1 = -1$. It is enough to show that $S(K_m^r \times G) = A(K_m^r) \times (S(G) + I) - I$. Writing A for $A(G)$ and $S(M) = -2M - I + J$ for any matrix M, We have to show that $S(J \otimes A) = J \otimes (S(A) + I) - I$. Observe that first,

$$S(B \otimes A) = -2(B \otimes A) - I \otimes I + J \otimes J$$

and so taking $B = J$ and moving $I \times I$ to the other side, $S(K_m^r \times G) + I$ is

$$S(J \otimes A) + I \otimes I = -2(J \otimes A) + J \otimes J = J \otimes (-2A + J) = J \otimes (S(A) + I). \quad \square$$

Lemma 2.3. *If the Seidel spectrum of G is $\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$ then the Seidel spectrum of $D_m^*(G)$ is*

$$Spec_s(D_m^*(G)) = \{m\sigma_i - (m-1) | i = 1, 2, \cdots, n\} \cup \{1^{mn-m}\}$$

Proof. Much like above,

$$\begin{aligned}[S(J \otimes (A+I) - I \otimes I] + I =& [-2(J \otimes (A+I) - I \otimes I + J \otimes J - I \otimes I] + I \otimes I \\ =& J \otimes (-2(A+I)) + J \otimes J \\ =& J \otimes (-2(A+I) + J) = J \otimes (S(A) - I). \qquad \square\end{aligned}$$

Theorem 2.4. *Let G be a graph with Seidel eigenvalues $\sigma_1, \sigma_2, \cdots, \sigma_n$ with $|\sigma_i| \geq \frac{m-1}{m}$, for all $1 \leq i \leq n$ then, $D_m(G)$ and $D_m^*(G)$ are Seidel non co-spectral equienergetic graphs if and only if G have equal numbers of positive and negative Seidel eigenvalues.*

Proof. It is not hard to see that the sums of the spectrums in these two lemma are the same. To say the same about the sums of the absolute values, (so that the Seidel energies are the same) we observe that we need that

$$\sum_i |m\sigma_i + (m-1)| = \sum_i |m\sigma_i - (m-1)|$$

Assuming that $|\sigma_i| > \dfrac{m-1}{m}$, we get that $|m\sigma_i + (m-1)| - |m\sigma_i - (m-1)|$ is $2(m-1)$ if σ_i is positive and $-2(m-1)$ if σ_i is negative for each $i = 1, 2, \cdots, n$. It follows that with these restrictions on the eigenvalues that the Seidel energies of $D_m(G)$ and $D_m^*(G)$ are the same if and only if G has the same number of positive and negative eigenvalues. This gives Theorem 2.4. $\square$

Observe that if G is a graph with n vertices then both $D_m(D_m^*(G))$ and $D_m^*(D_m(G))$ are graphs with m^2n vertices. We next prove following theorem for Seidel equienergetic graphs.

Theorem 2.5. *Let G be a graph with Seidel eigenvalues $\sigma_1, \sigma_2, \cdots, \sigma_n$ with $|\sigma_i| \geq \left(\frac{m-1}{m}\right)^2$, for all $1 \leq i \leq n$ then, $D_m^*(D_m(G))$ and $D_m(D_m^*(G))$ are Seidel non co-spectral equienergetic graphs if and only if G has equal numbers of positive and negative Seidel eigenvalues.*

Proof. By Lemma 2.2, $D_m(G)$ has spectrum $\{m\sigma_i + (m-1) |\, i = 1, 2, \cdots, n\} \cup \{-1^{mn-n}\}$ and by Lemma 2.3 spectrum of $D_m^*(D_m(G))$ is $\{m^2\sigma_i + (m-1)^2 |\, i = 1, 2, \cdots, n\} \cup \{(1-2m)^{mn-n}\} \cup \{1^{m^2n-mn}\}$ and again by Lemma 2.2 and Lemma 2.3, spectrum of $D_m(D_m^*(G))$ is $\{m^2\sigma_i - (m-1)^2 |\, i = 1, 2, \cdots, n\} \cup \{(2m-1)^{mn-n}\} \cup \{-1^{m^2n-mn}\}$. To prove $D_m^*(D_m(G))$ and $D_m(D_m^*(G))$ are equienergetic, it is enough to prove

$$\sum_i \left|m^2\sigma_i + (m-1)^2\right| = \sum_i \left|m^2\sigma_i - (m-1)^2\right|.$$

If $|\sigma_i| > \dfrac{(m-1)^2}{m^2}$, we have $\left|m^2\sigma_i + (m-1)^2\right| - \left|m^2\sigma_i - (m-1)^2\right|$ is $2(m-1)^2$ if σ_i is positive and $-2(m-1)^2$ if σ_i is negative. It follows that $D_m^*(D_m(G))$ and $D_m(D_m^*(G))$ are Seidel equienergetic if and only if G has equal numbers of positive and negative Seidel eigenvalues. □

3. Concluding Remarks

The concept of Seidel equienergetic graphs is analogous to the concepts of equienergetic graphs. We present here methods to construct Seidel equienergetic graphs by means of G^r and G^{ur} from a given graph G.

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