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Spectra of Graphs Obtained by Duplication of Graph Elements

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Abstract

The concept of graph energy is a frontier between two important branches of basic sciences: namely, Mathematics and Chemistry. The sum of absolute values of eigenvalues of adjacency matrix of graph is called the energy of a graph. The graph energy of some standard graphs is available in literature while we have investigated the energy of a graphs which are obtained from the duplication of graph elements (Vertex and Edge) in a given graph.

Keywords: Duplication, Eigenvalue, Energy of Graphs

AMS Subject Classification(2010): 05C50, 05C76

1 Introduction

All the graphs considered here are simple, finite, connected and undirected. For standard terminology and notations related to graph theory we follow West [2] while any terms related to algebra we rely upon Lang [12].

The *adjacency matrix* $A(G)$ of G is defined as $A(G) = [a_{ij}]$, where $a_{ij} = 1$ if v_i is adjacent with v_j, and 0 otherwise. The characteristic polynomial of the adjacency matrix of G is the characteristic polynomial of G, denoted by $\phi(G : x)$. The roots of the equation $\phi(G : x) = 0$, denoted by $\lambda_1, \lambda_2, \cdots, \lambda_n$ are said to be eigenvalues of G and their collection is the spectrum of G. Hence,

$$spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ m_1 & m_2 & \cdots & m_n \end{pmatrix}$$

The energy of a graph G is the sum of absolute values of the eigenvalues of graph G and denoted by $E(G)$. Hence,

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

The concept of graph energy was introduced by Gutman [4] in 1978.

In chemistry the eigenvalues represents the energy levels of the electrons in a molecule of a conjugated hydrocarbon. A conjugated hydrocarbon can be represented by a graph called molecular graph in which every carbon atom is represented by a vertex, carbon-carbon bond by an edge and hydrogen atoms are ignored. The study of molecular structure with the help of energy of its graph is now termed as chemical graph theory. The total π-electron energy E is the sum of the energies of all electrons in a molecule. The total π-electron energy is same as the energy of molecule graph. Therefore, to investigate the energy of a graph is equivalent to find the total π-electron energy of a molecule.

A brief account of energy of graph can be found in Cvetkoviċ [3] and Li [13]. The concepts like Incidence energy [5], Skew energy [1], Distance energy [9] are also introduced in recent past.

Two non-isomorphic graphs G_1 and G_2 of the same order are said to be *equienergetic* if $E(G_1) = E(G_2)$. Two graphs are said to be co-spectral if they have same spectra. It is always challanging to find out non co-spectral equienergetic graphs because co-spectral graphs are obviously equienergetic. Balakrishnan [8] have proved that for any positive integer $n \geq 3$, there exist non co-spectral, equienergetic graphs of order $4n$.

The *m- splitting graph* $Spl_m(G)$ of a graph G is obtained by adding to each vertex v of G new m vertices, say $v_1, v_2, v_3, \cdots, v_m$ such that v_i, $1 \leq i \leq m$ is adjacent to each vertex that is adjacent to v in G.

The *m-shadow graph* $D_m(G)$ of a connected graph G is constructed by taking m copies of G, say $G_1, G_2, \cdots, G_m$, then join each vertex u in G_i to the neighbors of the corresponding vertex v in G_j, $1 \leq i, j \leq m$.

Vaidya and Popat [10, 11] have proved that $E(Spl_m(G)) = \sqrt{1+4m}E(G)$ and $E(D_m(G)) = mE(G)$ as well as $Spl_2(G)$ and $D_3(G)$ are non-cospectral equienergetic .

A graph G is $k-$ *regular graph* if for some positive integer k, $d(v) = k$, for each vertex v of the graph G.

Let G be a graph with n vertices, e edges, and no self loops. *The incidence matrix* $B(G)$ of graph G is an $n \times e$ matrix $[b_{ij}]$ such that $b_{ij} = 1$, if j^{th} edge e_j is incident on i^{th} vertex v_i, and 0 otherwise.

Let $A \in R^{m\times n}$, $B \in R^{p\times q}$. Then the *Kronecker product* (or tensor product) of A and B is defined as the matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

Proposition 1.1. [7] If $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ be invertible matrices then

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$$

Proposition 1.2. [7, 6] Let $M, N, P, Q \in \mathbb{R}^{n\times n}$ be matrices, Q be invertible and

$$S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}$$

then, $det\ S = det\ Q\ .\ det\ \left[M - NQ^{-1}P\right]$

2 Main Results

Definition 2.1. *Duplication of a vertex* v_k by a new edge $e = v'v''$ in a graph G produces a new graph G_1 such that $N(v') = \{v_k, v''\}$ and $N(v'') = \{v_k, v'\}$.

Theorem 2.2. Let G be a graph with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and G_1 be the graph obtained from G by duplicating each vertex of G by a new edge then

$$E(G_1) = n + \sum_{\lambda_i \le 2} \sqrt{\lambda_i^2 - 2\lambda_i + 9} + \sum_{\lambda > 2} (\lambda_i + 1)$$

Proof: Let $v_1, v_2, \cdots, v_n$ be the vertices of a graph G then the adjacency matrix $A(G)$ is given by

$$A(G) = \begin{array}{c} \\ \mathbf{v_1} \\ \mathbf{v_2} \\ \mathbf{v_3} \\ \vdots \\ \mathbf{v_n} \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} & \cdots & \mathbf{v_n} \end{array} \\ \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix} \end{array}$$

We duplicate the vertices $v_1, v_2, \cdots, v_n$ all together by the edges $e_1, e_2, \cdots, e_n$ respectively such that, $e_1 = v_1'v_1''$, $e_2 = v_2'v_2''$, $\cdots$, $e_n = v_n'v_n''$ to obtain graph G_1
The adjacency matrix of G_1 is given by in terms of block matrix as follow

$$A(G_1) = \begin{array}{c|ccccc:ccccccc} & v_1 & v_2 & \cdots & v_n & v_1' & v_1'' & v_2' & v_2'' & \cdots & v_n' & v_n'' \\ \hline v_1 & 0 & a_{12} & \cdots & a_{1n} & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ v_2 & a_{21} & 0 & \cdots & a_{2n} & 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_n & a_{n1} & a_{n2} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ \hdashline v_1' & 1 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ v_1'' & 1 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ v_2' & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ v_2'' & 0 & 1 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_n' & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ v_n'' & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{array}$$

Let $B = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}$

Then,

$$A(G_1) = \begin{bmatrix} A(G) & B \\ B^T & I_n \otimes A(K_2) \end{bmatrix}$$

The characteristic polynomial of above matrix is given by

$$\begin{aligned}
\phi(G_1 : x) &= |xI_{3n} - A(G_1)| \\
&= \begin{vmatrix} xI_n - A(G) & B \\ B^T & I_n \otimes (xI_2 - A(K_2)) \end{vmatrix} \\
&= |I_n \otimes (xI_2 - A(K_2))||xI_n - A(G) - B(I_n \otimes (xI_2 - A(K_2)))^{-1}B^T| \\
&= (x^2-1)^n|xI_n - A(G) - B((xI_2 - A(K_2))^{-1} \otimes I_n^{-1})B^T| \\
&= (x^2-1)^n\left|xI_n - A(G) - B\left(\frac{1}{x^2-1}(xI_2 + A(K_2)) \otimes I_n\right)B^T\right| \\
&= |(x^2-1)(xI_n - A(G)) - B((xI_2 + A(K_2)) \otimes I_n)B^T|
\end{aligned}$$

Now,

$$B((xI_2 + A(K_2)) \otimes I_n)B^T = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix} \begin{bmatrix} x & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & x & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & x & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & x \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} x+1 & x+1 & 0 & 0\cdots & 0 & 0 \\ 0 & 0 & x+1 & x+1\cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0\cdots & x+1 & x+1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2x+2 & 0 & 0 & \cdots & 0 \\ 0 & 2x+2 & 0 & \cdots & 0 \\ 0 & 0 & 2x+2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 2x+2 \end{bmatrix}$$

$$= (2x+2)I_n$$

Continuing proof of theorem

$$\begin{aligned}\phi(G_1 : x) &= |(x^2-1)(xI_n - A(G)) - B((xI_2 + A(K_2)) \otimes I_n)B^T| \\ &= |(x^2-1)(xI_n - A(G)) - (2x+2)I_n)|\end{aligned}$$

It follows that if $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of A then

$$\begin{aligned}\phi(G_1 : x) &= \prod_{i=1}^{n} \left[(x^2-1)(x-\lambda_i) - (2x+2)\right] \\ &= \prod_{i=1}^{n} [(x-1)(x+1)(x-\lambda_i) - 2(x+1)] \\ &= (x+1)^n \prod_{i=1}^{n} \left(x^2 - x\lambda_i - x + \lambda_i - 2\right)\end{aligned}$$

The roots of above characteristic polynomial are

$$x = -1(n\,\text{times}),\ x = \frac{(\lambda_i+1) \pm \sqrt{\lambda_i^2 - 2\lambda_i + 9}}{2}$$

For each $i = 1, 2, \cdots, n$
Hence,

$$spec(G_1) = \begin{pmatrix} -1 & \frac{(\lambda_1+1)+\sqrt{\lambda_1^2-2\lambda_1+9}}{2} & \cdots & \frac{(\lambda_n+1)+\sqrt{\lambda_n^2-2\lambda_n+9}}{2} \\ n & 1 & \cdots & 1 \\ \frac{(\lambda_1+1)-\sqrt{\lambda_1^2-2\lambda_1+9}}{2} & \frac{(\lambda_2+1)-\sqrt{\lambda_2^2-2\lambda_2+9}}{2} & \cdots & \frac{(\lambda_n+1)-\sqrt{\lambda_n^2-2\lambda_n+9}}{2} \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

The calculation of energy considers only positive eigenvalues. For the graph under consideration, the positivity of eigenvalues depends upon the value $(\lambda_i + 1) - \sqrt{\lambda_i^2 - 2\lambda_i + 9}$. This give rise to following two possibilities.

$$\begin{aligned}(\lambda_i+1) &\leq \sqrt{\lambda_i^2 - 2\lambda_i + 9} && \text{if } \lambda_i \leq 2 \\ (\lambda_i+1) &> \sqrt{\lambda_i^2 - 2\lambda_i + 9} && \text{if } \lambda_i > 2\end{aligned}$$

Here,

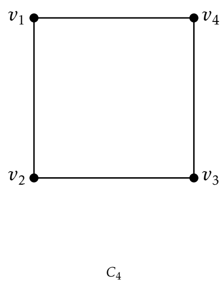
$$\begin{aligned}E(G_1) &= \sum_{i=1}^{3n} |\lambda_i| \\ &= \sum_{i=1}^{n} |-1| + \sum_{i=1}^{n} \left|\frac{(\lambda_i+1)+\sqrt{\lambda_i^2-2\lambda_i+9}}{2}\right| + \sum_{i=1}^{n} \left|\frac{(\lambda_i+1)-\sqrt{\lambda_i^2-2\lambda_i+9}}{2}\right|\end{aligned}$$

$$= n + \sum_{\lambda_i \le 2}^{n} \left[\frac{(\lambda_i + 1) + \sqrt{\lambda_i^2 - 2\lambda_i + 9}}{2} + \frac{\sqrt{\lambda_i^2 - 2\lambda_i + 9} - (\lambda_i + 1)}{2} \right]$$

$$+ \sum_{\lambda_i > 2}^{n} \left[\frac{(\lambda_i + 1) + \sqrt{\lambda_i^2 - 2\lambda_i + 9}}{2} + \frac{(\lambda_i + 1) - \sqrt{\lambda_i^2 - 2\lambda_i + 9}}{2} \right]$$

$$= n + \sum_{\lambda_i \le 2} \sqrt{\lambda_i^2 - 2\lambda_i + 9} + \sum_{\lambda_i > 2} (\lambda_i + 1)$$

Illustration 2.3. Consider cycle C_4 and a graph (say G_1) obtained from C_4 by duplicating each vartex by an edge. It is obvious that $E(C_4) = 4$ as spec(C_4)=$\begin{pmatrix} -2 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$



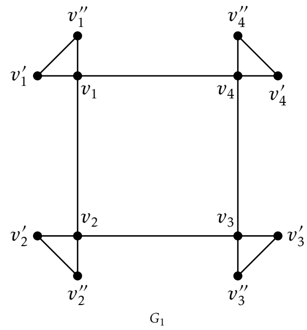


Figure 1:

$$A(G_1) = \begin{array}{c|cccccccccccc} & v_1 & v_2 & v_3 & v_4 & v_1' & v_1'' & v_2' & v_2'' & v_3' & v_3'' & v_4' & v_4'' \\ \hline v_1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_2 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ v_3 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ v_4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ v_1' & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_1'' & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_2' & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ v_2'' & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ v_3' & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ v_3'' & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ v_4' & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ v_4'' & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

Therefore,

$$spec(G_1) = \begin{pmatrix} \frac{-1+\sqrt{17}}{2} & \frac{-1-\sqrt{17}}{2} & 3 & 2 & -1 & 0 \\ 1 & 1 & 1 & 2 & 6 & 1 \end{pmatrix}$$

The following table compares spectrum of C_4 and G_1

Table 1:

spectrum of C_4	**spectrum of** $G_1 = \frac{(\lambda_i+1)\pm\sqrt{\lambda^2-2\lambda+9}}{2}$
$\lambda_1 = -2$	$\frac{-1+\sqrt{17}}{2}, \frac{-1-\sqrt{17}}{2}$
$\lambda_2 = 2$	0, 3
$\lambda_3 = 0$	2, -1

Definition 2.4. *Duplication of an edge* $e = v_iv_{i+1}$ by a vertex v' in a graph G produces a new graph G_1 such that $N(v') = \{v_i, v_{i+1}\}$.

Theorem 2.5. Let G be a $k-$ regular graph with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and G_1 be the graph obtained from G by duplicating each edge of G by a new vertex then

$$E(G_1) = \sum_{i=1}^{n} \sqrt{\lambda_i^2 + 4(\lambda_i + k)}$$

Proof: Let $v_1, v_2, \cdots, v_n$ be the vertices and $e_1, e_2, \cdots, e_m$ be the edges of k-regular graph G then the adjacency matrix $A(G)$ and incidence matrix $X(G)$ are given by

$$A(G) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \vdots \\ \boldsymbol{v_n} \end{array} \begin{array}{c} \begin{array}{ccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \cdots & \boldsymbol{v_n} \end{array} \\ \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix} \end{array}, B(G) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \vdots \\ \boldsymbol{v_n} \end{array} \begin{array}{c} \begin{array}{ccccc} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} & \cdots & \boldsymbol{e_m} \end{array} \\ \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1m} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2m} \\ b_{31} & b_{32} & b_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nm} \end{bmatrix} \end{array}$$

We duplicate the edges $e_1, e_2, \cdots, e_m$ all together by the vertices $e'_1, e'_2, \cdots, e'_m$ respectively to obtained a graph G_1

The adjacency matrix of G_1 is given by in terms of block matrix as follow

$$A(G_1) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \vdots \\ \boldsymbol{v_n} \\ \boldsymbol{e'_1} \\ \boldsymbol{e'_2} \\ \boldsymbol{e'_3} \\ \vdots \\ \boldsymbol{e'_m} \end{array} \begin{array}{c} \begin{array}{ccccccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \cdots & \boldsymbol{v_n} & \boldsymbol{e'_1} & \boldsymbol{e'_2} & \boldsymbol{e'_3} & \cdots & \boldsymbol{e'_m} \end{array} \\ \left[\begin{array}{cccc|ccccc} 0 & a_{12} & \cdots & a_{1n} & b_{11} & b_{12} & b_{13} & \cdots & b_{1m} \\ a_{21} & 0 & \cdots & a_{2n} & b_{21} & b_{22} & b_{23} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 & b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nm} \\ \hline b_{11} & b_{21} & \cdots & b_{n1} & 0 & 0 & 0 & \cdots & 0 \\ b_{12} & b_{22} & \cdots & b_{n2} & 0 & 0 & 0 & \cdots & 0 \\ b_{13} & b_{23} & \cdots & b_{n3} & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{1m} & b_{2m} & \cdots & b_{nm} & 0 & 0 & 0 & \cdots & 0 \end{array}\right] \end{array}$$

That is,

$$A(G_1) = \begin{bmatrix} A(G) & B(G) \\ B(G)^T & O_n \end{bmatrix}$$

The characteristic polynomial of above matrix is given by

$$\begin{aligned} \phi(G_1 : x) &= |xI_{n+m} - A(G_1)| \\ &= \begin{vmatrix} xI_n - A(G) & B(G) \\ B(G)^T & xI_m \end{vmatrix} \\ &= |xI_m||xI_n - A(G) - B(G)(xI_m)^{-1}B(G)^T| \\ &= x^m\left|xI_n - A(G) - \frac{1}{x}B(G)B(G)^T\right| \\ &= x^{m-n}|x^2I_n - xA(G) - (A + kI_n)| \end{aligned}$$

It follows that if $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of A then

$$\phi(G_1 : x) = x^{m-n}\prod_{i=1}^{n}\left(x^2 - x\lambda_i - (\lambda_i + k)\right)$$

The roots of above characteristic polynomial are

$$x = 0(m - n\,\text{times}),\ x = \frac{\lambda_i \pm \sqrt{\lambda_i^2 + 4(\lambda_i + k)}}{2}$$

For each $i = 1, 2, \cdots, n$
Hence,

$$spec(G_1) = \begin{pmatrix} 0 & \frac{\lambda_1 + \sqrt{\lambda_1^2 + 4(\lambda_1 + k)}}{2} & \cdots & \frac{\lambda_n + \sqrt{\lambda_n^2 + 4(\lambda_n + k)}}{2} & \frac{\lambda_1 - \sqrt{\lambda_1^2 + 4(\lambda_1 + k)}}{2} & \cdots & \frac{\lambda_n - \sqrt{\lambda_n^2 + 4(\lambda_n + k)}}{2} \\ m - n & 1 & \cdots & 1 & 1 & \cdots & 1 \end{pmatrix}$$

For Any eigenvalue λ of $k-$ regular graph

$$\begin{aligned} & -k \leq \lambda \leq k \\ &\Rightarrow \lambda \geq -k \\ &\Rightarrow \lambda + k \geq 0 \\ &\Rightarrow 4(\lambda + k) \geq 0 \\ &\Rightarrow \lambda^2 - (\lambda^2 + 4(\lambda + k)) \leq 0 \\ &\Rightarrow \lambda^2 \leq (\lambda^2 + 4(\lambda + k)) \\ &\Rightarrow \lambda \leq \sqrt{(\lambda^2 + 4(\lambda + k))} \end{aligned}$$

Now,

$$E(G_1) = \sum_{i=1}^{n} \left| \frac{\lambda_i + \sqrt{\lambda_i^2 + 4(\lambda_i + k)}}{2} \right| + \sum_{i=1}^{n} \left| \frac{\lambda_i - \sqrt{\lambda_i^2 + 4(\lambda_i + k)}}{2} \right|$$
$$= \sum_{i=1}^{n} \left(\frac{\lambda_i + \sqrt{\lambda_i^2 + 4(\lambda_i + k)}}{2} + \frac{\sqrt{\lambda_i^2 + 4(\lambda_i + k)} - \lambda_i}{2} \right)$$
$$= \sum_{i=1}^{n} \sqrt{\lambda_i^2 + 4(\lambda_i + k)}$$

Illustration 2.6. Consider cycle C_4 and a graph (say G_1) obtained from C_4 by duplicating each edge by vertex. It is obvious that $E(C_4) = 4$ as spec(C_4)=$\begin{pmatrix} -2 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$

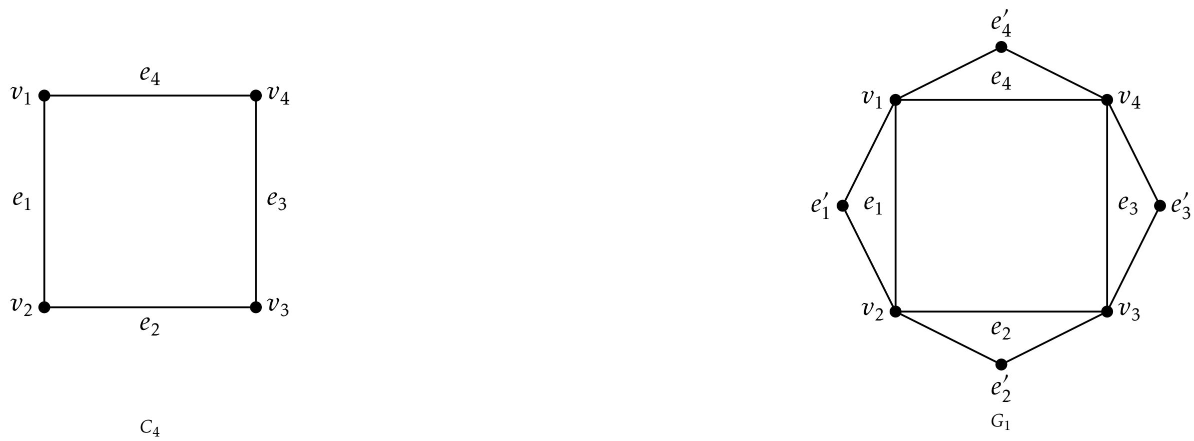


Figure 2:

$$A(G_1) = \begin{array}{c} \\ \mathbf{v_1} \\ \mathbf{v_2} \\ \mathbf{v_3} \\ \mathbf{v_4} \\ \mathbf{e'_1} \\ \mathbf{e'_2} \\ \mathbf{e'_3} \\ \mathbf{e'_4} \end{array} \begin{array}{c} \begin{array}{cccccccc} \mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3} & \mathbf{v_4} & \mathbf{e'_1} & \mathbf{e'_2} & \mathbf{e'_3} & \mathbf{e'_4} \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Therefore,

$$spec(G_1) = \begin{pmatrix} 1 + \sqrt{5} & 1 - \sqrt{5} & -\sqrt{2} & \sqrt{2} & -2 & 0 \\ 1 & 1 & 2 & 2 & 1 & 1 \end{pmatrix}$$

The following table compares spectrum of C_4 and G_1

Table 2:

spectrum of C_4	**spectrum of G_1 = $\frac{\lambda \pm \sqrt{\lambda^2+4(\lambda+k)}}{2}$**
$\lambda_1 = -2$	-2, 0
$\lambda_2 = 2$	$1+\sqrt{5}$, $1-\sqrt{5}$
$\lambda_3 = 0$	$\sqrt{2}$, $-\sqrt{2}$

3 Concluding Remarks

This work is an effort to obtain the energy of a graph which is a supergraph of a given graph. To construct a supergraph we consider the duplication of graph elements. We have investigated the energy of graphs obtained by duplication of vertex by edge as well as edge by vertex. We have found that it is possible to express the energy of newly constructed supergraph in terms of eigenvalues of the graph under consideration.

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