South East Asian J. of Mathematics and Mathematical Sciences
Vol. 20, Proceedings (2022), pp. 93-100

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

SOME NEW CLASSES OF EQUIENERGETIC GRAPHS

Kalpesh M. Popat

Department of Computer Applications,
Atmiya University,
Rajkot - 360005, Gujarat, INDIA

E-mail : kalpeshmpopat@gmail.com

(**Received:** *Apr. 08, 2022* **Accepted:** *Jul. 31, 2022* **Published:** *Aug. 30, 2022)*

Special Issue
Proceedings of National Virtual Conference on
"Emerging Trends in Discrete Mathematics, NCETDM - 2022"

Abstract: The eigenvalue of a graph G is the eigenvalue of its adjacency matrix and the energy $E(G)$ of graph G is the sum of absolute values of its eigenvalues. Two non-isomorphic graphs G_1 and G_2 of the same order are said to be equienergetic if they have same energies. The complement of a graph G is the graph $\overline{G}$ with vertex set $V(G) = V(\overline{G})$ and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. In the present work three pairs of equienergetic graphs have been obtained using graph complement.

Keywords and Phrases: Eigenvalue, Energy of Graph, Equienergetic Graphs.

2020 Mathematics Subject Classification: 05C50, 05C76.

1. Introduction and Preliminaries

All the graphs considered here are simple, finite and undirected. For standard terminology and notations related to graph theory we follow Balakrishnan and Ranganathan [2] while for the concept related to algebra, we follow Lang [7]. Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$.

The adjacency matrix $A(G)$ of a graph G with vertices $v_1, v_2, \cdots, v_n$ is an $n \times n$ matrix $[a_{ij}]$ such that,

$$\begin{aligned} a_{ij} &= 1, \text{ if } \ v_i \ \text{ is adjacent with } \ v_j \\ &= 0, \ \text{ otherwise} \end{aligned}$$

The eigenvalues of adjacency matrix of graph G are known as eigenvalues of graph. The collection of eigenvalues of the graph with their multiplicities is known as spectrum of the graph. If $\lambda_1, \lambda_2, \cdots, \lambda_k$ are the distinct eigenvalues of G with respective multiplicities $m_1, m_2, \cdots, m_k$, then the spectrum of G is denoted by,

$$spec(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ m_1 & m_2 & \cdots & m_k \end{pmatrix}, \text{where } \sum_{i=1}^{k} m_i = n$$

Two non-isomorphic graphs are said to be cospectral if they have the same spectra, otherwise they are known as non-cospectral. The energy $E(G)$ of a graph G is the sum of absolute values of the eigenvalues of graph G.
Hence,

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

The concept of energy was introduced by Gutman [6]. A brief account of energy of graph can be found in Cvetković *et al.* [5] and Li *et al.* [8]. Two non-isomorphic graphs G_1 and G_2 of same order are said to be *equienergetic* if $E(G_1) = E(G_2)$. Obviously, co-spectral graphs are always equienergetic. Balakrishnan and Ranganathan [2] showed the existence of non-cospectral equienergetic graphs. In 2005 Stevanović [12] constructed eqienergetic graphs of order $p \equiv 0(mod 5)$. A systematic computer aided study have been carried out for equienergetic trees by Brankov *et al.* [3] and Milijkoić *et al.* [9]. Vaidya *et al.* [13, 14] have obtained some new classes of equienergetic graph using various graph operations.

The complement of a graph G is the graph $\overline{G}$ with vertex set $V(G) = V(\overline{G})$ and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. A graph G with $G \cong \overline{G}$ is called self- complementary graph. Recently, Ramane *et al.* [10] have obtained non self-complementary graphs for which $E(G) = E(\overline{G})$. Such graphs are known as *complementary equienergetic graphs.* By the computer aided search Akabar Ali *et al.* [1] investigated complementary equienergetic graphs of order at most 10.

Proposition 1.1. [11] *Let G be an r-regular graph of order n with the eigenvalues $r, \lambda_1, \lambda_2, \cdots, \lambda_n$. Then the eigenvalues of $\overline{G}$ are $n - r - 1, -\lambda_2 - 1, \cdots, -\lambda_n - 1$.*

Definition 1.2. *The Cartesian product of graphs G and H is a graph, denoted as $G \times H$, whose vertex set is $V(G) \times V(H)$. Two vertices (u_1, v_1) and (u_2, v_2) in $G \times H$ are adjacent if $u_1 = u_2$ and v_1 and v_2 are adjacent in H or $v_1 = v_2$ and u_1 and u_2 are adjacent in G.*

Definition 1.3. *The Kronecker product of G and H is denoted by $G \otimes H$ with vertex set $V(G) \times V(H)$ and two vertices(u_1, v_1) and (u_2, v_2) in $G \otimes H$ are adjacent if and only if u_1 and u_2 are adjacent in G as well as v_1 and v_2 are adjacent in H.*

Proposition 1.4. [2] *If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of G and $\mu_1, \mu_2, \cdots, \mu_m$ are the eigenvalues of H, then*

i the eigenvalues of $G \times H$ are $\lambda_i + \mu_j$, $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, m$.

ii the eigenvalues of $G \otimes H$ are $\lambda_i \mu_j$, $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, m$.

Definition 1.5. *The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G say G' and G''. Join each vertex u' in G' to the neighbors of the corresponding vertex u'' in G''.*

Definition 1.6. *The extended shadow graph $D_2^*(G)$ of a connected graph G is constructed by taking two copies of G say G' and G''. Join each vertex u' in G' to the neighbours of the corresponding vertex u'' and with u'' in G''.*

If G is a graph of order n then $D_2(G)$ and $D_2^*(G)$ are graphs of order $2n$ and, if G is an r-regular graph then $D_2(G)$ and $D_2^*(G)$ are also regular graphs with degrees $2r$ and $2r + 1$ respectively.

Proposition 1.7. [13] *If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of G then $2n$ eigenvalues of $D_2(G)$ are $2\lambda_1, 2\lambda_2, \cdots, 2\lambda_n, \quad 0\,(n\,times)$.*

Proposition 1.8. [14] *If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of G then $2n$ eigenvalues of $D_2^*(G)$ are $2\lambda_1 + 1, 2\lambda_2 + 1, \cdots, 2\lambda_n + 1, -1(n\,times)$.*

Definition 1.9. *Let G be a graph with the vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$. The extended bipartite double graph, $Ebd(G)$ of a graph G is the bipartite graph with its partite sets $X = \{x_1, x_2, \cdots, x_n\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$ in which two vertices x_i and y_j are adjacent if $i = j$ or v_i and v_j are adjacent in G.*

If G is a graph of order n then $Ebd(G)$ is of order $2n$ and, if G is an r-regular graph then $Ebd(G)$ is an $(r + 1)$-regular graph.

Proposition 1.10. [4] *Let G be a graph of order n with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then the eigenvalues of extended bipartite double graph $Ebd(G)$ are $\pm(\lambda_1+1), \pm(\lambda_2+1), \cdots \pm (\lambda_n + 1)$.*

2. Main Results

Theorem 2.1. *If $G \ncong K_n$ is such that regular graph then*

$$E(\overline{Ebd(G)}) = E(\overline{G \times K_2}) = E(\overline{\overline{G} \otimes K_2})$$

Proof. Let G be an r-regular graph with eigenvalues $r, \lambda_2, \lambda_2, \cdots, \lambda_n$. Therefore, by Proposition 1.1 the eigenvalues of $\overline{G}$ are $n-r-1, -\lambda_2-1, \cdots, -\lambda_n-1$. According to Proposition 1.10 and Proposition 1.4, the spectra of $Ebd(G)$, $G \times K_2$ and $\overline{G} \otimes K_2$ are respectively

$$Spec(Ebd(G)) = \begin{pmatrix} r+1 & \lambda_2+1 & \cdots & \lambda_n+1 & -(r+1) & -(\lambda_2+1) & \cdots & -(\lambda_n+1) \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$Spec(G \times K_2) = \begin{pmatrix} r+1 & \lambda_2+1 & \cdots & \lambda_n+1 & r-1 & \lambda_2-1 & \cdots & \lambda_n-1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$Spec(\overline{G} \otimes K_2) = \begin{pmatrix} n-r-1 & -\lambda_2-1 & \cdots & -\lambda_n-1 & -n+r+1 & \lambda_2+1 & \cdots & \lambda_n+1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

Again, by Proposition 1.1

$$Spec(\overline{Ebd(G)}) = \begin{pmatrix} 2n-r-2 & -\lambda_2-2 & \cdots & -\lambda_n-2 & r & \lambda_2 & \cdots & \lambda_n \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$Spec(\overline{G \times K_2}) = \begin{pmatrix} 2n-r-2 & -\lambda_2-2 & \cdots & -\lambda_n-2 & -r & -\lambda_2 & \cdots & -\lambda_n \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$Spec(\overline{\overline{G} \otimes K_2}) = \begin{pmatrix} n+r & \lambda_2 & \cdots & \lambda_n & n-r-2 & -\lambda_2-2 & \cdots & -\lambda_n-2 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

Thus,

$$E(\overline{Ebd(G)}) = E(\overline{G \times K_2}) = E(\overline{\overline{G} \otimes K_2}) = 2n-2+\sum_{i=2}^{n}(|\lambda_i|+|\lambda_i+2|)$$

Theorem 2.2. *If G is an $r-$ regular graph then*

$$E(\overline{D_2^*(G)}) = E(\overline{G} \otimes K_2)$$

Proof. Let G be a regular graph with eigenvalues $r, \lambda_2, \lambda_2, \cdots, \lambda_n$. Therefore, by Proposition 1.1 the eigenvalues of $\overline{G}$ are $n-r-1, -\lambda_2-1, -\lambda_3-1, \cdots, -\lambda_n-1$.

According to Proposition 1.4 and Proposition 1.8, the spectra of $\overline{G}\otimes K_2$ and $D_2^*(G)$ are respectively,

$$Spec(\overline{G}\otimes K_2) = \begin{pmatrix} n-r-1 & -\lambda_2-1 & \cdots & -\lambda_n-1 & -(n-r-1) & \lambda_2+1 & \cdots & \lambda_n+1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

and

$$Spec(D_2^*(G)) = \begin{pmatrix} 2r+1 & 2\lambda_2+1 & \cdots & 2\lambda_n+1 & -1 \\ 1 & 1 & \cdots & 1 & n \end{pmatrix}$$

Therefore, by Proposition 1.1 the spectrum of $\overline{D_2^*(G)}$ is,

$$Spec(\overline{D_2^*(G)}) = \begin{pmatrix} 2n-2r-2 & -2\lambda_2-2 & \cdots & -2\lambda_n-2 & -2\lambda_n-2 & 0 \\ 1 & 1 & \cdots & 1 & 1 & n \end{pmatrix}$$

Hence,

$$E(\overline{D_2^*(G)}) = 2n-2r-2+2\sum_{i=2}^{n}|\lambda_i+1| = E(\overline{G}\otimes K_2)$$

Remark 2.3. *In [08], it was proved that $E(D_2) = 2E(G)$. Thus, If G is self complementary euienergetic graph then $E(G) = E(\overline{G})$.*

$$\Rightarrow r+\sum_{i=2}^{n}|\lambda_i| = n-r-1+\sum_{i=2}^{n}|\lambda_i+1|$$

In this case, $E(\overline{D_2^*(G)}) = E(\overline{G}\otimes K_2) = 2(E(G)) = E(D_2(G))$

Theorem 2.4. *If G is an $r-$ regular graph with n vertices then*

$$E(\overline{Ebd(\overline{G})}) = E(\overline{\overline{G}\times K_2})$$

Proof. Let G be a regular graph with eigenvalues $r, \lambda_2, \cdots, \lambda_n$. Therefore, by Proposition 1.1 the eigenvalues of $\overline{G}$ are $n-r-1, -\lambda_2-1, \cdots, -\lambda_n-1$. By Proposition 1.10 and Proposition 1.4, spectra of $Ebd(\overline{G})$ and $\overline{G}\times K_2$ are respectively,

$$Spec(Ebd(\overline{G})) = \begin{pmatrix} n-r & -\lambda_2 & \cdots & -\lambda_n & -(n-r) & \lambda_2 & \cdots & \lambda_n \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$Spec(\overline{G}\times K_2) = \begin{pmatrix} n-r & -\lambda_2 & \cdots & -\lambda_n & n-r-2 & -\lambda_2-2 & \cdots & -\lambda_n-2 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

Thus, by Proposition 1.1,

$$Spec(\overline{Ebd(\overline{G})}) = \begin{pmatrix} n+r-1 & \lambda_2-1 & \cdots & \lambda_n-1 & n-r-1 & -\lambda_2-1 & \cdots & -\lambda_n-1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

and

$$Spec(\overline{\overline{G} \times K_2}) = \begin{pmatrix} n+r-1 & \lambda_2-1 & \cdots & \lambda_n-1 & -n+r+1 & \lambda_2+1 & \cdots & \lambda_n+1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

Now,

$$E(\overline{\overline{G} \otimes K_2}) = 2n-2+\sum_{i=2}^{n}|\lambda_i-1|+\sum_{i=2}^{n}|\lambda_i+1| = E(\overline{Ebd(\overline{G})})$$

3. Conclusion

The concept of graph energy have drawn attention of many researchers due to its applications in chemistry. We have investigated some new families equienergetic graphs using graph complement.

Acknowledgement

The present work is a part of the research work carried out under seed money Project No: SL/SMFAP/Phase 1/2021/007, dated: 27th Oct, 2021 by Atmiya University-Rajkot (Gujarat), India.

References

[1] Ali A., Elumalai S., Mansour T. and Rostami M. A., On the complementary equienergetic graphs, MATCH Commun. Math. Comput. Chem., 83 (2020), 555-570.

[2] Balakrishnan R. and Ranganathan K., A Textbook of Graph Theory, Springer, New York, 2000.

[3] Brankov V., Stevanoviĉ D. and Gutman I., Equienergetic chemical tree, J. Serb. Chem. Soc., 69 (2004), 549-553.

[4] Chen Z., Spectra of extended double cover graphs, Czechoslovak Math. J., 54 (4) (2004), 1077-1082.

[5] Cvetkoviĉ D., Rowlison P. and Simiĉ S., An Introduction to the Theory of Graph Spectra, Cambridge University press, 2010.

[6] Gutman I., The energy of a graph, Ber. Math. Statist. Sekt. Forschungsz, 103 (1978), 1-22.

[7] Lang S., Algebra, Springer, New York, 2002.

[8] Li X., Shi Y. and Gutman I., Graph Energy, Springer, New York, 2012.

[9] Milijkoič O., Furtula B., Radenkovič S. and Gutman I., Equienergetic and almost equienergetic trees, MATCH Commun. Math. Comput. Chem., 61 (2009), 451-461.

[10] Ramane H. S., Parvathalu B., Patil D. D. and Ashoka K., Graphs equienergetic with their complements, MATCH Commun. Math. Comput. Chem., 82 (2019), 471-480.

[11] Sachs H., Über selbstkomplementare Graphen, Publ. Math. Debrecen, 9 (1962), 270–288.

[12] Stevanovićc D., Energy and NEPS of graphs, Lin. Multilin. Algebra, 23 (2005), 67-74.

[13] Vaidya S. K. and Popat K. M., Some new results on energy of graphs, MATCH Commun. Math. Comput. Chem., 77 (2017), 589-594.

[14] Vaidya S. K. and Popat K. M., Some borderenergetic and equienergetic graphs, Kragujevac J. Math., 48 (2024), 935-949.

[15] Vaidya S. K. and Popat K. M., On equienergetic, hyperenergetic and hypoenergetic graphs, Kragujevac J. Math., Vol. 44 (4) (2020), 523-532.