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On Restrained Domination Number of Graphs

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Abstract

For a graph $G = (V, E)$, a set $S \subseteq V$ is a restrained dominating set if every vertex not in S is adjacent to a vertex in S and to a vertex in $V - S$. The smallest cardinality of a restrained dominating set of G is called restrained domination number of G, denoted by $\gamma_r(G)$. We investigate restrained domination number of some cycle related graphs which are obtained by means of various graph operations on cycle.

Keywords: Dominating set, restrained dominating set, restrained domination number.
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1 Introduction

We consider finite, connected and undirected graph $G = (V, E)$ without loops and multiple edges. The minimum degree among the vertices of graph G is denoted by $\delta(G)$ while the maximum degree among the vertices of graph G is denoted by $\triangle(G)$.

A set $S \subseteq V$ is a dominating set if every vertex $v \in V - S$ is adjacent to a vertex in S. A γ-set is a dominating set of minimum cardinality. The domination number $\gamma(G)$ is a minimum cardinality of a γ-set. A brief account of dominating sets and its related concepts can be found in Haynes *et al* [9]. Some variants of domination models such as total domination [4], equitable domination [12], global domination [11], independent domination [3, 10] are worth to mention. The present work is focused on one such variant known as restrained domination. A set $S \subseteq V$ is a restrained dominating set if every vertex not in S is adjacent to a vertex in S as well as to a vertex in $V - S$. The minimum cardinality of a restrained dominating set S is called the restrained domination number of G which is denoted by $\gamma_r(G)$. It is obvious that all mutually non-adjacent vertices must belong

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to every restrained dominating set. The concept of restrained domination was introduced by Telle and Proskurowski [13] as a vertex partitioning problem. The restrained domination in trees are well studied in [5, 8] while the graphs with minimum degree two are explored in the context of restrained domination by Domke *et al* [7]. The restrained domination in path, cycle, complete graph and multipartite graphs is discussed by Domke *et al* [6]. In the present work we investigate restrained domination number of some cycle related graphs.

We begin the next section by starting existing results and definitions needed for the advancement of the discussion.

2 Main Results

Definition 2.1. The switching of a vertex v of G means removing all the edges incident to v and adding edges joining v to every vertex which is not adjacent to v in G. We denote the resultant graph by $\widetilde{G}$.

Definition 2.2. The square of a graph G denoted by G^2 has the same vertex set as of G and two vertices are adjacent in G^2 if they are at distance 1 or 2 apart in G.

Definition 2.3. The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G say G' and G''. Join each vertex u' in G' to the neighbours of the corresponding vertex u'' in G''.

Definition 2.4. [1] The m-shadow graph $D_m(G)$ of a connected graph G is constructed by taking m copies of G, say $G_1, G_2, \cdots, G_m$, then join each vertex u in G_i to the neighbors of the corresponding vertex v in G_j, $1 \leq i, j \leq m$.

Definition 2.5. The splitting graph $S'(G)$ of a graph G is obtained by adding to each vertex v a new vertex v', such that v' is adjacent to every vertex that is adjacent to v in G.

Definition 2.6. [1] The m-splitting graph $Spl_m(G)$ of a graph G is obtained by adding to each vertex v of G new m vertices, say $v_1, v_2, v_3, \cdots, v_m$ such that v_i, $1 \leq i \leq m$ is adjacent to each vertex that is adjacent to v in G.

Proposition 2.7. [6] $\gamma_r(K_{1,n-1}) = n$, for $n \geqslant 2$.

Proposition 2.8. [2] $\gamma(C_n^2) = \left\lceil \frac{n}{5} \right\rceil$, for $n \geqslant 3$.

Theorem 2.9. $\gamma_r(\widetilde{C_n}) = \begin{cases} 4 & ;\text{for } n = 4 \\ 3 & ;\text{for } n \geqslant 5 \end{cases}$.

Proof: Let $v_1, v_2, ..., v_n$ be the vertices of C_n. Without loss of generality, we switch the vertex v_1. Then $V(\widetilde{C_n}) = V(C_n)$. To prove the result we consider the following two cases.

Case i: For $n = 4$

In this case $\widetilde{C_4} = K_{1,3}$. Then by Proposition 2.7 $\gamma_r(K_{1,3}) = \gamma_r(\widetilde{C_4}) = 4$.

Case ii: For $n \geqslant 5$

In this case the pendant vertices v_2, v_n are non-adjacent which must belong to every restrained dominating set S. Moreoever $\triangle(\widetilde{C_n}) = n - 3 = d(v_1)$. In order to attain the minimum cardinality, a restrained dominating set should contain the vertex v_1. For $5 \leqslant n \leqslant 8$, $S = \{v_1, v_2, v_n\}$ is a restrained dominating set of minimum cardinality, while for $n > 8$, $S = \{v_1, v_2, v_n\}$ is the only restrained dominating set of $\widetilde{C_n}$. That is, $|S|$ is minimum. Hence $\gamma_r(\widetilde{C_n}) = 3$, for $n \geqslant 5$. ■

Illustration 2.10. The switching of vertex v_1 of C_8 is shown in Figure 1 where the set of solid vertices is its restrained dominating set of minimum cardinality.

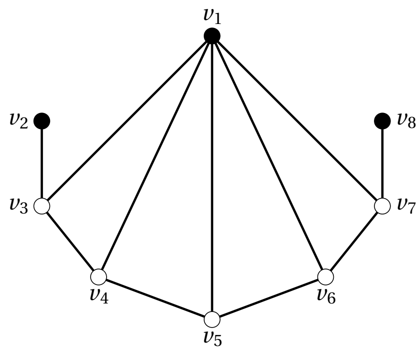

Figure 1: $\gamma_r(\widetilde{C_8}) = 3$

Theorem 2.11. $\gamma_r(C_n^2) = \left\lceil \frac{n}{5} \right\rceil$.

Proof: Let $\{v_1, v_2, ..., v_n\}$ be the vertices of C_n. Then $|V(C_n^2)| = |V(C_n)| = n$. Now by Proposition 2.8 $\gamma(C_n^2) = \left\lceil \frac{n}{5} \right\rceil$, we consider γ-set $S = \{v_{5k+1};\ k = 0, 1, 2, ..., \left\lceil \frac{n}{5} \right\rceil - 1\}$. Here $V - S = \{v_{5k+2}, v_{5k+3}, v_{5k+4}, v_{5k+5} /\ k = 0, 1, 2, 3, ..., \left\lceil \frac{n}{5} \right\rceil - 1\}$. Then S is a restrained dominating set because every vertex of $V - S$ is adjacent to at least one vertex of $V - S$. Now S being a γ-set, it is a restrained dominating set of minimum cardinality. Hence $\gamma_r(C_n^2) = \left\lceil \frac{n}{5} \right\rceil$. ■

Illustration 2.12. The square of cycle C_{10} is shown in Figure 2 where the set of solid vertices is its restrained dominating set of minimum cardinality..

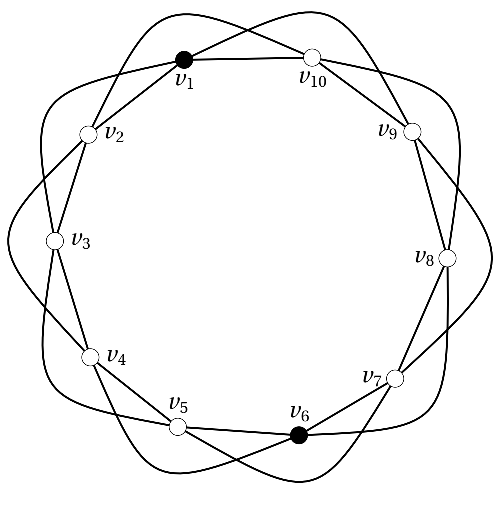

Figure 2: $\gamma_r(C_{10}^2) = 2$

Theorem 2.13. $\gamma_r(D_m(C_n)) = \begin{cases} \dfrac{n}{2} & ;\text{for } n \equiv 0(\text{mod}\, 4) \\ \left\lfloor \dfrac{n}{2} \right\rfloor + 1 & ;\text{otherwise} \end{cases}.$

Proof: Let $C_n^1, C_n^2, C_n^3, ..., C_n^m$ be the m copies of C_n and $u_i^1, u_i^2, u_i^3, ..., u_i^m$ $(1 \leqslant i \leqslant n)$ be the vertices of i^{th} copy C_n^i. Note that $D_m(C_n)$ is a $2m$-regular graph with $|D_m(C_n)| = mn$.

Let S_1 be a restrained dominating set. In order to attain the minimum cardinality, it should contain at least one neighbour of every vertex of any C_n^i $(1 \leqslant i \leqslant m)$. Hence $|S_1| \geqslant \dfrac{n}{2}$.

Consider a set $S \subseteq V(D_m(C_n))$ as follows,

$$S = \begin{cases} \{u_1^1, u_{4i}^1, u_{4i+1}^1, u_n^1\} & ;\text{if } n \equiv 0(\text{mod}\, 4) \\ \{u_1^1, u_{4j}^1, u_{4j+1}^1\} & ;\text{if } n \equiv 1(\text{mod}\, 4) \\ \{u_1^1, u_{4j}^1, u_{4j+1}^1, u_n^1\} & ;\text{if } n \equiv 2,\ 3(\text{mod}\, 4) \end{cases}.$$

where $1 \leqslant i < \dfrac{n}{4}$ and $1 \leqslant j \leqslant \left\lfloor \dfrac{n}{4} \right\rfloor$.

Then, $|S| = \begin{cases} \dfrac{n}{2} & ;\text{if } n \equiv 0(\text{mod}\, 4) \\ \left\lfloor \dfrac{n}{2} \right\rfloor + 1 & ;\text{if } n \equiv 1,\ 2,\ 3(\text{mod}\, 4) \end{cases}.$

Since every vertex not in S is adjacent to a vertex in S and to a vertex in $V - S$, it follows that S is a restrained dominating set of $D_m(C_n)$.

As $|S_1| \geqslant \dfrac{n}{2}$ for every restrained dominating set and $|S| = \dfrac{n}{2}$ for $n \equiv 0(\text{mod}\, 4)$, implies that S is a restrained dominating set of minimum cardinality. Hence $\gamma_r(D_m(C_n)) = \dfrac{n}{2}$, where $n \equiv 0(\text{mod}\, 4)$.

For $n \equiv 1,\ 2,\ 3(\text{mod}\, 4)$, if possible suppose that S' is a restrained dominating set such that $|S'| = \left\lfloor \dfrac{n}{2} \right\rfloor < |S|$. Now $\triangle(D_m(C_n)) = 2m$ and in order to attain the minimum cardinality, S' can not contain the vertices where each vertex among them can dominate distinct $2m$ vertices of $D_m(C_n)$. Moreover $\left\lfloor \dfrac{n}{2} \right\rfloor . \triangle\, (D_m(C_n)) + \left\lfloor \dfrac{n}{2} \right\rfloor < mn = |V(D_m(C_n))|$ for $n \equiv 1$ or $3(\text{mod}\, 4)$. Therefore S' can not be a restrained dominating set of $D_m(C_n)$ (not even a dominating set). This implies that S is a restrained dominating set of minimum cardinality for $n \equiv 1$ or $3(\text{mod}\, 4)$. Also for $n \equiv 2(\text{mod}\, 4)$ there are at most $\left\lfloor \dfrac{n}{3} \right\rfloor$ vertices in $D_m(C_n)$ are such that each of them can dominate $2m$ distinct vertices. Therefore from the adjacency nature of vertices of $D_m(C_n)$, it is obvious that $\left\lfloor \dfrac{n}{2} \right\rfloor$ vertices of S are not enough to dominate all the vertices of $D_m(C_n)$ for $n \equiv 2(\text{mod}\, 4)$. This implies that S' is not a restrained dominating set of $D_m(C_n)$. Hence, S is a restrained dominating set of $D_m(C_n)$ with minimum cardinality.

Thus, $\gamma_r(D_m(C_n)) = \begin{cases} \dfrac{n}{2} & ;\text{for } n \equiv 0(\text{mod}\, 4) \\ \left\lfloor \dfrac{n}{2} \right\rfloor + 1 & ;\text{otherwise.} \end{cases}$ ■

Illustration 2.14. 4-shadow graph of cycle C_6 is shown in Figure 3 where the set of solid vertices is its restrained dominating set of minimum cardinality.

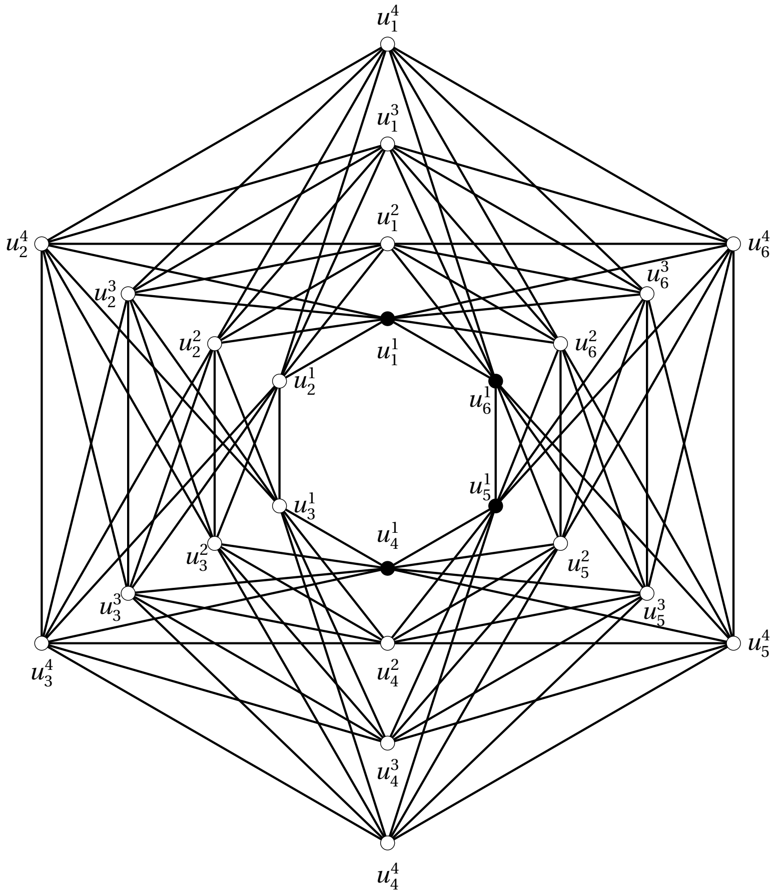

Figure 3: $\gamma_r(D_4(C_6) = 4$

Theorem 2.15. $\gamma_r\ (Spl_m(C_n)) = \dfrac{n}{2}$; for $n \equiv 0(\text{mod}\ 4)$.

Proof: Let $u_1, u_2, ..., u_n$ be the vertices of C_n and $u_1^j, u_2^j, u_3^j, \cdots, u_m^j$ with $(1 \leqslant j \leqslant m)$ be the vertices corresponding to $u_1, u_2, ..., u_n$ in $Spl_m(C_n)$. Then $|Spl_m(C_n)| = n(m+1)$. Here $\triangle(Spl_m(C_n)) = 2(m+1) = d(u_i)$ $(1 \leqslant i \leqslant n)$. In order to attain the minimum cardinality, every restrained dominating set S should contain at least one neighbour of every vertex u_i $(1 \leqslant i \leqslant n)$ of C_n. Therefore $|S| \geqslant \dfrac{n}{2}$.

Consider a set $S \subseteq V(Spl_m(C_n))$ as $S = \{u_1, u_{4i}, u_{4i+1}, u_n\}$ where $1 \leqslant i < \dfrac{n}{4}$. Then $|S| = \dfrac{n}{2}$. Since every vertex not in S is adjacent to a vertex in S and to a vertex in $V - S$, it follows that the set S is a restrained dominating set of $Spl_m(C_n)$.

If possible suppose that S' is a restrained dominating set such that $|S'| = \left(\dfrac{n}{2} - 1\right) < \dfrac{n}{2} = |S|$. Now $\triangle(Spl_m(C_n)) = 2(m+1) = d(u_i)$ $(1 \leqslant i \leqslant n)$. In order to attain the minimum cardinality, S'

can not contain the vertices where each vertex can dominate distinct $2(m+1)$ vertices of $Spl_m(C_n)$. Moreover $\left(\frac{n}{2}-1\right). \triangle\,(Spl_m(C_n)) + \left(\frac{n}{2}-1\right) = \frac{1}{2}.(2m+3)(n-2) < n(m+1) = |V(Spl_m(C_n))|$. Therefore S' can not be a restrained dominating set (even not a dominating set). This implies that S is a restrained dominating set of minimum cardinality $\frac{n}{2}$. Hence $\gamma_r(Spl_m(C_n)) = \frac{n}{2}$, for $n \equiv 0(\text{mod}\ 4)$.

■

Illustration 2.16. 2-splitting graph of cycle C_4 is shown in Figure 4 where the set of solid vertices is its restrained dominating set of minimum cardinaity.

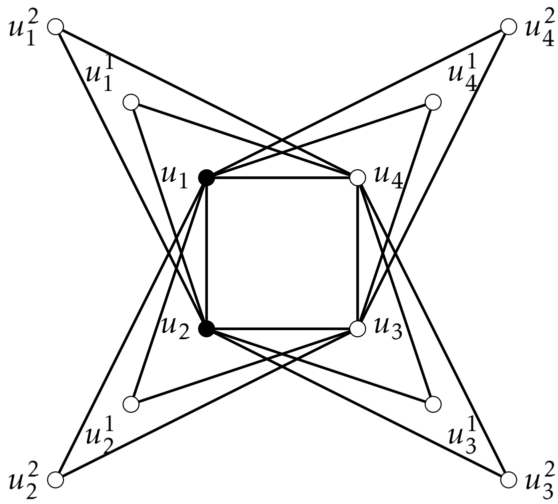

Figure 4: $\gamma_r(Spl_2(C_4)) = 2$

3 Concluding Remarks

The concept of restrained domination in graph is very important and interesting as well because it also takes into account the adjacency within the complement of a dominating set. We have obtained exact value of restrained domination number of larger graphs obtained from cycle by means of some graph operations.

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